Game Problems for Fractional Quasilinear Systems

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Abstract

The game problem of approaching of a cylindrical terminal set by a trajectory of dynamic fractional system is studied. The matrix version of these functions naturally appears in studies of the games, described by differential equations with derivatives of fractional order. Two classes of such games are treated. For games of the first class the evolution is described by Riemann-Liouville system of fractional equations [1], besides Riemann-Liouville fractional integral [1] of respective order is given at the initial instant of time. The dynamics of games, belonging to the second class, is subject to a system of equations with Dzhbrashyan-Nersesyan regularized fractional derivatives of [3, 4] together with the usual Cauchy conditions given at the initial instant of time. Under the assumption on the fulfilment of Pontryagin’s condition sufficient conditions for the games termination in finite time are derived on the basis of the method of resolving functions [5,7,8]. An example of the game problem with simple matrix is given which manifests the efficiency of suggested approach [9].

1 Problem statement

Let $A$ be a square matrix of order $n$ with real constant coefficients, $\beta \in (0,1)$, and let $\left( I_{0+}^{\beta} \Psi \right) (t)$ and $\left( D_{0+}^{\beta} \Psi \right) (t)$ be, respectively, Riemann-Liouville fractional integral and fractional derivative of order $\beta$ [1]

$$\left( I_{0+}^{\beta} \psi \right) (t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{1-\beta}} d\tau,$$

$$\left( D_{0+}^{\beta} \psi \right) (t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\psi(\tau)}{(t-\tau)^\beta} d\tau$$

Below the formula

$$\left( D^{(\beta)} \Psi \right) (t) = (\Gamma(1-\beta))^{-1}$$

$$\left[ \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} \Psi(\tau) d\tau - \Psi(+0)t^{-\beta} \right]$$

defines the regularized Dzhbrashyan-Nersesyan fractional derivative of order $\beta$ [3, 4]. The block of control is given by continuous function $\varphi(u,v); \varphi : U \times V \rightarrow \mathbb{R}^n$, where $U$ and $V$ are compacts in $\mathbb{R}^n$. The evolution of a process in the first problem is described by the system of differential equations

$$D^{(\beta)} \dot{z} = Az + \varphi(u,v)$$

under the initial condition

$$I_{0+}^{1-\beta} z|_{t=0} = z_0$$

while in the second one by the system of differential equations

$$D^{(\beta)} z = Az + \varphi(u,v)$$

under the initial Cauchy conditions

$$z|_{t=0} = z_0$$

Alongside with the conflict-controlled processes (1), (2) and (3),(4) the terminal set $M^*$, having the form

$$M^* = M_0 + M$$

is given, where $M_0$ is a linear subspace in $\mathbb{R}^n$ and $M$ is a compact from the orthogonal complement $L$ to $M_0$ in $\mathbb{R}^n$.

The goal of the first player ($u$) is to bring a trajectory of the process (1), (2) or the process (3), (4) to set (5) under any counteraction of the second player ($v$). In so doing the first
player is allowed to use information on the initial state of the process (1), (2) or (3), (4) and on the control prehistory of the second player [5] that is

\[ u(t) = u(z, v_1(t)), \quad \text{where} \]

\[ v_1(t) = \{v(s) : s \in [0, t]\}, \quad t > 0 \]

The second player uses as controls arbitrary Lebesgue measurable time-functions with values in \( V \).

2 Auxiliary results

Let us proceed to the deduction of integral representation of functions \( z(t) \) and \( \hat{z}(t) \). For this purpose for arbitrary positive \( r \) and arbitrary complex \( m \) we define the modified matrix function of Mittag-Leffler by the formula

\[ \hat{B} = B - \frac{B^k}{(kp^{-1} + \mu)} \]

where \( B \) is an arbitrary square matrix of order \( n \) with complex-valued elements. Note that \( \hat{B} \) is an integral matrix function of argument \( B \).

Lemma. For any measurable controls \( u(t) \in U, v(t) \in V \) the following assertions hold:

1. function \( \hat{z}(t) \) may be presented in the form

\[ \hat{z}(t) = \hat{g}(t)\hat{z}_0 + \int_0^t \hat{\Theta}(t - \tau)\varphi(u(\tau), v(\tau)) \, d\tau \]

where

\[ \hat{g}(t) = \hat{\Theta}(t) = \sum_{k=0}^{\infty} B^k(1 - (kp^{-1} + \mu)) \]

2. function \( z(t) \) may be presented by the formula

\[ z(t) = g(t)z_0 + \int_0^t \Omega(t - \tau)\varphi(u(\tau), v(\tau)) \, d\tau, \]

where

\[ g(t) = E_{1/\beta}(At^\beta; 1), \quad \Omega(t) = t^{\beta - 1}E_{1/\beta}(At^\beta; \beta). \]

The proof readily follows from formula (6). Note that if \( B = \lambda I \), where \( \lambda \) is a real number and \( I \) a unit matrix, then \( \hat{E}_\mu(B; \mu) = \hat{E}_\mu(\lambda; \mu)I \), where \( \hat{E}_\mu(\lambda; \mu) \) is the modified function of Mittag-Leffler [2]. Correspondingly, \( \hat{g}(t) = \hat{\Theta}(t) = \Omega_0(t)I, \quad g(t) = g_0(t)I, \quad \Omega_0(t) = t^{\beta - 1}E_{1/\beta}(At^\beta; \beta), \quad g_0(t) = E_{1/\beta}(\lambda t^\beta; 1). \)

3 Sufficient condition for the game termination

On the basis of representation (8) of a trajectory of the process (3), (4) we now deduce sufficient conditions for the solvability of the problem of approaching the method of resolving functions. In the case of the process (1), (2) the reasoning is similar.

Denote by \( \pi \) the orthoprojector, acting from \( R^n \) into \( L \), and consider the set-valued mappings

\[ W(t, v) = \pi\Omega(t)\varphi(U, v), W(t) = \bigcap_{v \in V} W(t, v) \]

Pontryagin’s Condition. The set-valued mapping \( W(t) \) is nonempty for all \( t \geq 0 \).

? From the results of [5–8] and the assumptions on parameters of the process (3), (4) there follows that mapping \( W(t) \) is Borel measurable. Therefore, at least a single Lebesgue measurable selection \( \gamma(t) \in W(t) \) exists. Fix it and denote

\[ \xi(t, z_0, \gamma(\cdot)) = \pi q(t)z_0 + \int_0^t \gamma(t - \tau) \, d\tau \]

For the game problem (3)-(5) we define the resolving function [5] by the formula

\[ \alpha(t, \tau, v) = \sup \{\alpha \geq 0 : [W(t - \tau, v) - \gamma(t - \tau)] \cap [M - \xi(t, z_0, \gamma(\cdot))] \neq \emptyset\} \]

and introduce with its help the set-valued mapping

\[ T(z_0, \gamma(\cdot)) = \left\{ t \geq 0 : \int_0^t \inf_{v \in V} \alpha(t, \tau, v) \, d\tau \geq 1 \right\} \]

The following result may be proved with the help of methods, developed in [5].

Theorem. Let Pontryagin’s condition hold in the game (3)-(5) and \( M = \text{co}M \).

Then, if for the initial state \( z_0 \) there exist a measurable selection \( \gamma(t) \in W(t) \) and a finite instant of time \( T > 0 \), such that

\[ T \in T(z_0, \gamma(\cdot)) \]

then a trajectory of the process (3), (4) can be brought from the initial state \( z_0 \) to terminal set (5) at the instant \( T \).

Proof. Consider the case \( \xi(T, z_0, \gamma(\cdot)) \in M \). Let \( u_T(\cdot) \) be an arbitrary measurable function with values in \( V \). Analogously to [5], we introduce a test function

\[ h(t) = 1 - \int_0^t \alpha(T, \tau, v(\tau)) \, d\tau, \quad t \in [0, T]. \]

Since the function \( \alpha(T, \tau, v) \) is \( L \times B \) measurable [9,10], it is superpositionally measurable as well, i.e. function
\( \alpha(T, \tau, v(\tau)) \) is measurable. On the other hand, by assumptions concerning the parameters of process (3), (5) the latter is bounded for almost all \( \tau < T \) and therefore integrable on only finite interval of time. It follows that the function \( h(t) \) is continuous, nonincreasing and \( h(0) = 1 \). Therefore, there exists an instant \( t_* = t(v(\cdot)), t_* \in (0, T) \) such that \( h(t_*) = 0 \).

In the sequel the segments \([0, t_*)\) and \([t_*, T]\) will be referred to as active and passive respectively. Let us describe how the first player chooses his control on each of them. For this purpose, consider a set-valued mapping

\[
U(\tau, v) = \{ u \in U : \pi \Omega(T - \tau) \phi(u, v) - \gamma(T - \tau) \in \alpha(T, \tau, v)| M - \xi(T, z_0, \gamma(\cdot)) \} \tag{9}
\]

Since the function \( \alpha(T, \tau, v) \) is \( L \times B \) measurable, \( M \) is a compact and the vector \( \xi(T, z_0, \gamma(\cdot)) \) is bounded, then the mapping \( \alpha(T, \tau, v)| M - \xi(T, z_0, \gamma(\cdot)) \) is \( L \times B \) measurable. In addition, it is obvious that the left side of inclusion in (9) is jointly \( L \times B \) measurable function in \( \tau \) and \( v \) and continuous in \( u \). From here, in view of the known statement from [11], it follows that the mapping \( U(\tau, v) \) is \( L \times B \) measurable. Therefore its selection

\[
u(\tau, v) = \text{lexmin} U(\tau, v)
\]

is \( L \times B \) measurable function [11]. On the active segment \([0, t_*)\) control of the first player is set equal to

\[
u(\tau) = u(\tau, v(\tau))
\]

By virtue of the function \( u(\tau, v) \) \( L \times B \) measurability, it is superpositionally measurable that implies the measurability of function \( u(\tau) \).

Let us analyze the passive segment \([t_*, T]\). We set in expression (9) \( \alpha(T, \tau, v) \equiv 0 \) for \( \tau \in [t_*, T] \), \( v \in V \), and choose control of the first player in accordance with the above outlined scheme using expressions (9)-(11).

In the case \( \xi(T, z_0, \gamma(\cdot)) \in M \) control of the first player is chosen from the same relations as on the passive segment, i.e. by the scheme (9)-(11) with \( \alpha(T, \tau, v) \equiv 0 \), \( \tau \in [0, T] \), \( v \in V \). Let us show that if the control of the first player is chosen in the form (11), then in both cases, in view of relations (9), (10), a trajectory of process (3) will be brought to set \( M \) at instant \( T \) for any control of the second player. From expression (3) we have

\[
\pi z(T) = \pi g(T)z_0 + \int_0^T \Omega(T - \tau)\phi(u(\tau), v(\tau)) d\tau \tag{12}
\]

Let us first analyze the case \( \xi(T, z_0, \gamma(\cdot)) \notin M \). To do this we add and subtract from the right side of equality (12) the term \( \int_0^T \gamma(\tau) d\tau \). Using the above outlined rule for control choice of the first player, we obtain from (3) the inclusion

\[
\pi z(T) \in \xi(T, z_0, \gamma(\cdot)) + \int_0^{t_*} \alpha(T, \tau, v(\tau)) d\tau + \int_0^{t_*} \alpha(T, \tau, v(\tau)) M d\tau
\]

Since \( M \) is a convex compact, \( \alpha(T, \tau, v(\tau)) \) is nonnegative function for \( \tau \in [0, t_*) \), and

\[
\int_0^{t_*} \alpha(T, \tau, v(\tau)) d\tau = 1,
\]

then \( \int_0^{t_*} \alpha(T, \tau, v(\tau)) M d\tau = M \) and, consequently, \( \pi z(T) \in M \) and \( z(T) \in M^* \).

Suppose that \( \xi(T, z_0, \gamma(\cdot)) \in M \). Then, taking into account the rule for control of the first player, from equality (12) one can immediately deduce the inclusion \( \pi z(T) \in M \).

Assertion. Let the problem (3)-(5) be linear \( \varphi(u, v) = v - v \) and Pontryagin’s condition hold.

Then, if a continuous nonnegative function \( r(t) \) and a number \( l \geq 0 \) exist such that

\[
\pi \Omega(t) U = r(t) S, \; M^* = l S
\]

where \( S \) is a ball of unit radius centred at the origin, then for \( \xi(t, z_0, \gamma(\cdot)) \in M \) the resolving function \( \alpha(t, \tau, v) \) appears as the greatest positive solution of the quadratic equation for \( \alpha \geq 0 \)

\[
\|\pi \Omega(t - \tau)v + \gamma(t - \tau) - \alpha \xi(t, z_0, \gamma(\cdot))\| = r(t - \tau) + al \tag{13}
\]

Thus, under the assumptions of assertion the construction of the resolving function is essentially facilitated. What is more, it becomes possible to evaluate the time of the game termination. Let us demonstrate this on the example with simple matrix for the games (1), (2), (5) and (3)-(5).

4 Example

Let \( A = \lambda I, \varphi(u, v) = u - v, \; M^* = \{0\}, \; U = aS, \; a > 1, \; V = S \).

Then \( L = \mathbb{R}^n, \; \pi = I \). The corresponding set-valued mappings have forms

\[
W(t, v) = \Omega_0(t)(aS - v), \; W(t) = |\Omega_0(t)| (a - 1) S
\]

and therefore Pontryagin’s condition holds.

Set \( \gamma(t - \tau) \equiv 0 \). Then

\[
\xi(t, z_0, 0) = \hat{g}(t)z_0, \; \xi(t, z_0, 0) = g(t)z_0
\]

Here \( l = 0, r(t) = a|\Omega_0(t)| \). The resolving functions may be found as the greatest root of corresponding equation (13). Then, using the reasoning and the computations, analogous to that, given in [5, 6], we come to the formulas

\[
\hat{T}(z_0) = \min \left\{ t \geq 0 : \Phi(t) \geq \frac{|z_0|}{(a - 1)} \right\}
\]
\[ T(z_0) = \min \left\{ t \geq 0 : |\Phi(t)| \geq \frac{||z_0||}{(a-1)} \right\}, \]  

where
\[
\hat{\Phi}(t) = \int_0^t \left| \Omega_0(t) \right| d\tau / \left| \Omega_0(t) \right|,
\]
\[
\Phi(t) = \int_0^t |\Omega_0(t)| d\tau / |\Omega_0(t)|
\]

If \( \lambda > 0 \) then \( \Omega_0(t) > 0 \) and the following equations are true [2]
\[
\hat{\Phi}(t) = tE_{1/\beta} (\lambda t^\beta; \beta + 1) / E_{1/\beta} (\lambda t^\beta; \beta);
\]
\[
\Phi(t) = tE_{1/\beta} (\lambda t^\beta; \beta + 1) / E_{1/\beta} (\lambda t^\beta; 1)
\]

Using the asymptotic (as \( t \to \infty \)) representations of the modified Mittag-Leffler functions [2] we infer that when
\[
||z_0|| < \lambda^{-1/\beta}(a-1), \ |z_0| < \lambda^{-1}(a-1)
\]
the time of approach the terminal set is finite and may be found as the least positive solution of the equations
\[
\hat{\Phi}(t) = ||z_0|| / (a-1), \ \Phi(t) = ||z_0|| / (a-1)
\]

If, otherwise, \( \lambda < 0 \) then functions \( E_{1/\beta} (\lambda t^\beta; \beta), 0 < \beta < 1 \), and \( E_{1/\beta} (\lambda t^\beta; 1) \) vanish as \( t \to \infty \) [2]. In view of the fact that the numerators in the expressions for \( \hat{\Phi}(t) \), \( \Phi(t) \) are bounded below, one can easily deduce that the processes (1), (2), (5) and (3)–(5) are completely conflict controlled [5].

For \( \lambda = 0 \)
\[
\hat{T}(\hat{z}_0) = \hat{\beta} \hat{z}_0 / a - 1, \ T(z_0) = \left( \frac{\Gamma(\beta + 1) ||z_0||}{a-1} \right)^{1/\beta}
\]

The above presented reasons and computations provide the foundation for the comparative computational analysis of the game termination time dependance on parameters \( \beta, \lambda, \hat{z}_0, z_0, a \). Note, that from formula (16), for example, there follows that for great positive \( \lambda \) the set of initial states, for which this time is finite, is larger for the problem (3)–(5) than for the problem (1), (2), (5) at the same value of parameter \( a \).

5 Conclusion

The suggested scheme is applicable to a wide range of functional-differential systems, in particular, integral, integral-differential and difference-differential systems of equations. The method may be successfully used in study of the game problems with groups of participants, with state constraints and with groups of participants, with state constraints and with the terminal functional.

References