Analysis of the Geometric Structure of the Hamilton-Jacobi Equation via Generating Functions of Symplectic Transforms

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Abstract

The geometric structure of the Hamilton-Jacobi equation is analyzed by using symplectic geometry. Generating function of symplectic transform plays an important role. It will be shown that the Hamilton-Jacobi equation possesses important geometric properties such as existence condition and maximality of the stabilizing solution, which are well-known in the Riccati equation.

1 Introduction

The Hamilton-Jacobi equation plays a fundamental role in nonlinear control theory. Solvability of many important problems, such as optimal regulation, $H^2$ and $H^\infty$ control problems, dissipative system theory and factorization theory, etc. is represented by that of the Hamilton-Jacobi equation and feedback functions depend on the solution of the Hamilton-Jacobi equation (see, e.g. [4, 5, 6]). Only a few researches are reported on the study of exact solutions of the Hamilton-Jacobi equation. In [13, 14] it is proved that if the Riccati equation constructed from the linear approximation of the Hamilton-Jacobi equation can be solved, then there exists, locally, a solution of the Hamilton-Jacobi equation and nonlinear $L^2$ gain is also discussed by the linearization argument. In [10, 11] it is explained how the solution method of the Hamilton-Jacobi equation can be reduced to the problem of ordinary differential equations using symplectic geometry. Also a necessary and sufficient condition for the existence of a stabilizing solution is presented in [10, 11].

In this paper we investigate the property and structure of the Hamilton-Jacobi equation by using symplectic geometry. Symplectic geometry plays a central role also in [13, 14], but the linearization argument is not employed in the present paper. Without linear approximation, it will be shown that major properties of the Riccati equation can be generalized and similar structures can be found in the Hamilton-Jacobi equation. The theory of generating functions of symplectic transforms [1, 7, 8] plays a central role to study the geometric structure of the Hamilton-Jacobi equation. Also, the structure of solutions, such as uniqueness of stabilizing solution and maximality and minimality of stabilizing and anti-stabilizing solutions, of the Hamilton-Jacobi equation is clarified.

2 Main Results

We investigate the Hamilton-Jacobi equation arising from nonlinear control theory of the form

\[(HJ) \quad H(x, p) = p^T f(x) - \frac{1}{2}p^TR(x)p + Q(x) = 0,\]

where \((x_1, \ldots, x_n)\) are independent variables, \(z(x_1, \ldots, z_n)\) is an unknown function, and \(p_1 = \partial z/\partial x_1, \ldots, p_n = \partial z/\partial x_n\). We assume that \(f : X \to \mathbb{R}^n, R : X \to \mathbb{R}^{n \times n}, Q : X \to \mathbb{R}\) are all \(C^\infty\), where \(X\) denotes the \(n\) dimensional space for \(x\), and that \(R(x)\) are symmetric for all \(x \in X\) and \(f\) and \(Q\) satisfy \(f(0) = 0, Q(0) = 0\) and \(\partial Q / \partial x(0) = 0\).

The following notion of stabilizing solution plays an important role in control theory.

**Definition 1** A solution \(z(x)\) of (HJ) is said to be a stabilizing solution if \(p(0) = 0\) and 0 is an asymptotically stable equilibrium of the vector field \(f(x) - R(x)p(x)\), where \(p(x) = (\partial z/\partial x)^T(x)\). Also, a solution \(z(x)\) is called an anti-stabilizing solution if \(p(0) = 0\) and \(-f(x) + R(x)p(x)\) has an asymptotically stable equilibrium at 0.

2.1 The Riccati and Hamilton-Jacobi Equations

The following theorem gives a necessary and sufficient condition for the existence of the stabilizing solution to (HJ), which is a natural extension of the well-known theory of the Riccati equation.
Theorem 2 [10, 11] There exists a stabilizing solution to 
(HJ) if and only if there exists an immersion $g$ from a neigh-
borhood of 0 in $\mathbb{R}^n$ into $T^*X$ with $g(0) = (0, 0)$ and 
g$\mathbb{R}^n \to X$ a diffeomorphism, and a vector field $X_A$ on 
$\mathbb{R}^n$ with asymptotically stable equilibrium $0 \in \mathbb{R}^n$ such that 
the condition 

$$X_H \circ g = Dg \cdot X_A$$

is satisfied in a neighborhood of 0 $\in \mathbb{R}^n$, where 
$X_H$ is the Hamiltonian vector field associated with 
$H$ and $g$ has the form $g(u) = (g_1(u), g_2(u)) = 
g((I, \cdots, I_n, u_1, \cdots, u_n))$, $g_2(u) \in T^*_{g_1(u)}X$, $g_1(u) \in X$. Under the above assumption, 
z$(x)$ is obtained from the integrable Pfaffian equation 

$$dz = g_{21} \circ g^{-1}_1(x)dx_1 + \cdots + g_{2n} \circ g^{-1}_1(x)dx_n.$$ 
If, in addition, $g_1$ is a global diffeomorphism on $\mathbb{R}^n$ and (1) holds for a globally asymptotically stable vector field $X_A$, then a global stabilizing solution is obtained from (2).

Let us consider the algebraic Riccati equation 

$$(\text{RIC}) \quad PA + A^TP - PRP + Q = 0,$$ 
where $A$ is a real square matrix of dimension $n$ and $R$ and $Q$ are symmetric matrices of dimension $n$. The $2n \times 2n$ matrix 

$$\text{Ham} = \begin{pmatrix} A & -R \\ -Q & -A^T \end{pmatrix}$$

is called the Hamiltonian matrix of (RIC). A necessary and 
sufficient condition for the existence of a stabilizing solution 
[2, 3, 9] is that (i) $\text{Ham}$ has no eigenvalues on the imagi-
ary axis, and (ii) the generalized eigenspace $E_-$ for $n$ stable 
eigenvalues satisfies the following complementary condition; 

$$E_- \oplus \text{Im} \begin{pmatrix} 0 \\ I \end{pmatrix} = \mathbb{R}^{2n}.$$ 
The proof is based on the following. Let $E_1$ and $E_2$ be the 
matrixes whose eigenvalues are the stable and unstable eigen-
vales of $\text{Ham}$, respectively. Then, there exist $n \times n$ matrix 
$T_1, T_2, T_3$ and $T_4$ satisfying 

$$\begin{pmatrix} T_1 & T_2 & T_3 & T_4 \end{pmatrix} \begin{pmatrix} A \\ -Q \\ -A^T \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$= T_1^T T_2 = T_2^T T_1$$

$$T_3^T T_4 = T_4^T T_3$$

$$T^T J T = J,$$

where 

$$T = \begin{pmatrix} T_1 & T_3 \\ T_2 & T_4 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$ 
We can also show that $E_2 = -E_2^T$.

Remark: It can be seen that Theorem 2, when applied to a 
Hamilton-Jacobi equation of the form 

$$(HJ)’ \quad H’(x, p) = p^T A x - \frac{1}{2} p^T R p + \frac{1}{2} x^T Q x = 0,$$

reduces to the above theory of the Riccati equation.

The following example motivates us to investigate the ge-
ometric structure of the Hamilton-Jacobi equation.

Example. Consider the equation 

$$H’ = -p_1 x_1 + p_2 (x_1 + x_2) - \frac{1}{2} p_2^2 + \frac{1}{2} x_1^2 = 0.$$ 
This is a Riccati equation for 

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 
We are going to solve (4) as a partial differential equation. Then, a solution 

$$P = \begin{pmatrix} 1/2 + c - c^2/2 & c \\ c & 0 \end{pmatrix}$$

is obtained from the first integral $p_2/x_1$ of $X_{H’}$, where $c$ is a non-zero constant. However, this solution is not a stabilizing 
one because $A - RP$ is not stable. To find a stabilizing solu-
tion, we need to find another integral. It is a simple but not an 
easy task to find out that 

$$\Omega(x, t) = \frac{1}{2} p_2 x_1 - \frac{1}{2} x_1^2 = 0,$$

is a first integral of $X_{H’}$. From this integral (with zero constant) 
and (4), we get 

$$P = \begin{pmatrix} 1 & 1 \\ 1/2 & 0 \end{pmatrix}.$$ 
The former is a stabilizing solution.

This illustrates the difficulty of finding all solutions as a 
partial differential equation even for the Riccati equation.

2.2 The geometric structure of the Hamilton-Jacobi 
equation

Let $\alpha : x_i’ = x_i'(x, p), p_i’ = p_i’(x, p) (i = 1, \cdots, n)$ be 
a symplectic transform. Then, for subsets $I = \{i_1, \cdots, i_l\}$ 
and $J = \{j_1, \cdots, j_{n-l}\}$ of $\{1, \cdots, n\}$ with $I \cap J = \phi$, it holds that 

$$\left| \frac{\partial(x_1’, \cdots, x_n’)}{\partial(x_{a-1}, x_{a}, \cdots, x_n, p_{i_1}, \cdots, p_{i_l})} \right| (0, 0) \neq 0,$$

and there exists a function $\Omega(x_1, x_n, p_I), (x_I = (x_{i_1}, \cdots, x_{i_l}), p_I = (p_{i_1}, \cdots, p_{i_l}))$ such that 

$$\sum_{k=1}^n p_k N_k - d\Omega = -\sum_{k=1}^n p_k dx_k'.$$

$\Omega$ is called a generating function of the symplectic transform 
$\alpha$. Lagrangian submanifolds are closely related to symplectic 
transforms, therefore, analysis of Lagrangian submanifolds 
can be replaced by that of generating functions.

The following theorem is an application of the theory of 
generating functions of symplectic transforms [1, 7, 8] and 
reveals the geometric structure of the Hamilton-Jacobi equa-

$$\sum_{k=1}^n p_k N_k - d\Omega = -\sum_{k=1}^n p_k dx_k'.$$
Corollary 5 Let $z(x)$ be a solution of (HJ) defined on a neighborhood of $0 \in X$ and $x'_k = p_k - p_k(x)$ for $1 \leq k \leq n$. Suppose there exists a solution $z''(x')$ around $x'$ to the auxiliary equation

$$x'^T f^*(p'') - \frac{1}{2} x'^T R(p'') x' = 0,$$

where $p''_k = \partial z''/\partial x_k'$ (1 \leq k \leq n) and $f^*(x) = f(x) - R(x)p(x)$. Then, together with $x'_k(x, p)$ (1 \leq k \leq n), the functions $p'_k(x, p), \ldots, p'_n(x, p)$ defined by

$$p'_k(x, p) = -x_k + \frac{\partial z''}{\partial x_k'}(x'(x, p)) = -x_k + \frac{\partial x''}{\partial x_k'}(x'(x, p)),$$

form a symplectic transform

$$\alpha: \begin{cases} x'_k = p_k - p_k(x) \\ p'_k = -x_k + \frac{\partial z''}{\partial x_k'}(x'(x, p)) \end{cases} (1 \leq k \leq n)$$

and the Hamiltonian vector field $X_H$ leaves the submanifold $\Lambda_+ = \{(x, p) | p'_1 = \cdots = p'_n = 0 \}$ invariant.

Theorem 4 Assume that the conditions of Theorem 3 are satisfied and $p''(0) = 0$. Then,

(i) $(\alpha \cdot X_H)(0, p') = \begin{pmatrix} 0 \\ -f^*(-p') \end{pmatrix}$, which equals

$$\begin{pmatrix} \frac{\partial f^*}{\partial x}(x) \\ 0 \end{pmatrix}$$

in $x$ coordinates.

(ii) $(\alpha \cdot X_H)(x', 0) = \begin{pmatrix} f^{**}(x') \\ 0 \end{pmatrix}$, where

$$f^{**}(x') = -\left(\frac{\partial f^*}{\partial x}(x')\right)^T (p''(x')) x' + \frac{1}{2} \left(\frac{\partial f^*}{\partial x}(x')\right)^T (p''(x')) x'.$$

Note that $\alpha \cdot X_H$ denotes the push-forward of $X_H$ by $\alpha$, that is,

$$(\alpha \cdot X_H)(x', p') = D\alpha(\alpha^{-1}(x', p')) X_H(\alpha^{-1}(x', p')).$$

Corollary 5 Under the assumptions in Theorem 4,

$$Df^{**}(0) = -Df^*(0)^T.$$  

Remark. Corollary 5 states that Theorem 4 generalizes the result $E_2 = -E_2^T$ in the theory of the Riccati equation.

Example (continued). The stabilizing solution of (4) can be obtained more easily by following the method in Theorem 3. The solution $P = \begin{pmatrix} 1/2 + c \cr c/2 \cr 0 \end{pmatrix}$ was not a stabilizing solution. Let $c = 1$ for the sake of simplicity. Then, $f^*(x) = (-x_1, x_2, x'_1 = p_1 - x_1 - x_2$ and $x'_2 = p_2 - x_1$. The auxiliary equation (5) is

$$-x'_1 p''_1 + x'_2 p''_2 - \frac{1}{2} x'_2^2 = 0.$$  

The first integral $x'_1 p''_1$ for the associated Hamiltonian vector field is found, and we have $p''_1 = 0$ and $p''_2 = \frac{1}{2} x'_2$. Thus, from (6),

$$p'_1 = -x_1 + 0 = -x_1$$

$$p'_2 = -x_2 + \frac{1}{2} x'_2 = \frac{1}{2} p_2 - \frac{1}{2} x_1 - x_2.$$  

Note that

$$dx'_1 \land dp_1 + dx'_2 \land dp_2 = (dp_1 - dx_1 - dx_2) \land (-dx_1)$$

$$+ (dp_2 - dx_1) \land \left(\frac{1}{2} dp_2 - \frac{1}{2} dx_1 - dx_2\right)$$

$$= dx_1 \land dp_1 + dx_2 \land dp_2$$

and the submanifolds $\{x'_1 = 0\}, \{x'_2 = 0\}, \{p'_1 = 0\}$, and $\{p'_2 = 0\}$ are invariant for the Hamiltonian vector field. The stabilizing solution is obtained from $\{x'_1 = p'_2 = 0\}$.

Next theorem concerns the structure of solutions of the Hamilton-Jacobi equation, which also extends the well-known properties of the Riccati equation to the Hamilton-Jacobi equation.

Theorem 6 Suppose that $R(x)$ is positive semi-definite for all $x$ in (HJ). Then, stabilizing and anti-stabilizing solutions, if they exist, are unique in a neighborhood of the origin. Let $z_1(x), z_2(x)$ and $z_3(x)$ be the stabilizing solution, an arbitrary solution with $p_2(0) = 0$ and the anti-stabilizing solution, respectively. Then, it follows that

$$z_3(x) \leq z_2(x) \leq z_1(x)$$

for all $x$ in a neighborhood of the origin.

This theorem is proved by using the following lemma.

Lemma 7 Let us consider the Lyapunov equation of the form

$$L = \sum_{i=1}^{n} f_i(x) \frac{\partial u}{\partial x_i} = Q(x),$$

and suppose that 0 is an asymptotically stable equilibrium of $f(\xi) = (f_1(\xi), \cdots, f_n(\xi))$ and $Q(\xi)$ satisfies $Q(0) = 0$. Denote $\xi(t, x)$ the solution of $\xi = f(\xi)$ starting from $x \in \mathbb{R}^n$ at $t = 0$.

(i) If $\int_0^\infty Q(\xi(t, x)) \, dt$ converges for all $x$ in a neighborhood of 0 and

$$\int_0^\infty \frac{\partial Q}{\partial \xi}(\xi(t, x)) \frac{\partial \xi(t, x)}{\partial x} \, dt$$

uniformly converges in a neighborhood of 0, then there exists a unique solution $u(x)$ of (L) that is defined around 0 and satisfies $u(0) = 0$, and it is given by

$$u(x) = -\int_0^\infty Q(\xi(t, x)) \, dt.$$

(7)
(ii) Conversely, if (L) has a solution \( u(x) \) with \( u(0) = 0 \), then for all \( x \) in a neighborhood of 0

\[
\int_0^\infty Q(\xi(t, x)) \, dt < \infty,
\]

and \( u(x) \) is given by (7).

3 Conclusion

In this paper, the geometric structure of the Hamilton-Jacobi equation in nonlinear control theory has been clarified using symplectic geometry. Generating function of symplectic transform plays an important role. The structure of solutions of the Hamilton-Jacobi equation has also been clarified and turned out to be a natural generalization of that of the Riccati equation.

Details and proofs are omitted in the present paper. They can be found in [12].

References


