About numerical approximations of a stabilizing distributed delay control law

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Abstract

Since the end of the seventies, several authors proposed to use a distributed delay in the control law of time-delay systems. Some recent publications showed that the implementation of such laws is not always safe with respect to the stability of the closed-loop system, as we recall in the first part of this paper limited to the scalar case of systems with delayed input. In the second part, we give a condition of stabilizability issued from complex analysis considerations. The last part of this work is devoted to the expression of conditions on the parameters of the system that match the condition given in the previous part. We exhibit one necessary and one sufficient condition of stabilizability for system with delayed input.

1 Motivation

In 1979, A. Manitius and A. Olbrot [2] proposed a method of pole placement for time-delay systems by using a distributed delay in the control law, i.e. a finite integral over the past values of the state or the command of the system. For example, a system with a pure delay on its input:

\[ \dot{x}(t) = ax(t) + u(t - h) \]  

would be controlled by a law of the form

\[ u(t) = f\dot{x}(t) + \int_0^h g(\tau)u(t - \tau)d\tau + v(t) \]  

where \( v \) is the input of the closed-loop system, \( f \) is a real, and \( g(\tau) \) is a function which equals zero for \( \tau \notin [0, h] \).

The transfer function of the system (1) is

\[ \frac{\dot{x}(s)}{u(s)} = \frac{e^{-sh}}{s - a} \]  

and the Laplace transform of the control law (2) is:

\[ \dot{u}(s) = f\dot{x}(s) + \dot{g}(s)\dot{u}(s) + \dot{v}(s) \]  

where \( \dot{x}(s) \) is the Laplace transform of the function \( x(t) \). Therefore, the transfer function of the closed-loop system is

\[ \frac{\dot{x}(s)}{\dot{u}(s)} = \frac{e^{-sh}}{(1 - \dot{g}(s))(s - a) - fe^{-sh}}. \]  

The strength of this method is that the function \( g(\tau) \) can be chosen so that the transfer function (5) has one pole whom value can be freely assigned [2]. On the other hand, this method rises the problem of the implementation of the distributed delay. One could think that it can be considered as a sum of two dynamics:

\[ \int_0^h g(\tau)u(t - \tau)d\tau = \int_t^{t-h} g(t - \theta)u(\theta)d\theta = \int_0^t g(t - \theta)u(\theta)d\theta - \int_0^{t-h} g(t - \theta)u(\theta)d\theta. \]

This leads to an internally unstable system if the system (1) is itself unstable (see [2]). Consequently, the implementation must respect the fact that the distributed delay is an integral over a constant interval. But the integral can not be exactly computed by numerical means. The authors above suggest to implement an approximation of the integral, calculated through a numerical quadrature. The integral is then replaced by a finite sum of punctual delays and the control law becomes:

\[ u(t) = fx(t) + vt + \sum_{n=0}^{N} \gamma_n \left( \frac{nh}{N} \right) u(t - \left( \frac{nh}{N} \right)) \]  

where \( \{\gamma_n\}_{n=0}^{N} \) are real and depend on the method chosen to approximate the integral. The Laplace transform of this control law is:

\[ \dot{u}(s) = f\dot{x}(s) + \dot{v}(s) + \left( \sum_{n=0}^{N} \gamma_n \left( \frac{nh}{N} e^{-\frac{n\pi i}{N}} \right) \right) \dot{u}(s) \]
In [5], it is shown that in the case of command law of the type (2), this procedure may lead to an unstable closed-loop system. Indeed, with this control law, the transfer of the system becomes:

\[
\frac{\hat{\xi}(s)}{\bar{v}(s)} = \frac{e^{-sh}}{(1 - \hat{\gamma}(s))(s - a) - fe^{-sh}} \quad (7)
\]

with

\[
\hat{\gamma}(s) = \sum_{n=0}^{N} \gamma_n \left( \frac{nh}{N} \right) e^{-s\frac{nh}{N}}.
\]

Viewing the denominator of the transfer as a polynomial in \( s \) with its coefficients in the ring of polynomials in \( e^{-s\frac{N}{N}} \), we see that the highest degree coefficient is the non-constant term \((1 - \hat{\gamma}(s))\). It follows that (7) is the transfer of a neutral system. Consequently, if this coefficient has unstable zeros, the system is itself unstable [1].

Also, a problem occurs when \((1 - \hat{\gamma}(s))\) has some unstable zeros. Indeed, \(\hat{\gamma}(s)\) is defined such that the transfer (5) (obviously non neutral in the spirit of the pole assignment) is stable, regardless of the locus of its zeros. So, if these last are unstable they are approximated by those of \((1 - \hat{\gamma}(s))\), what have for consequence, as we write above, that the transfer (7) is unstable. An example of such a system theoretically stable with a law of the form (2) but unstable with an approximation of the form (6) is given in [5].

So, the following is concerned with the determination of the class of control laws of the type (6) such that the closed-loop system (7) turns out to be stable.

### 2 A condition of stabilizability

When we replace the distributed delay by a numerical quadrature (with constant step), the denominator of the closed-loop system is

\[
(s - a) \left[ 1 - \sum_{n=0}^{N} \alpha_n \exp\left(-\frac{nh}{N}\right) \right] + \Delta \exp((a - s)h),
\]

where \(\Delta\) is the value of the denominator at \(a\).

Let us denote by \(P(z)\) the polynomial \(1 - \sum_{n=0}^{N} \alpha_n z^n\) which is not to vanish at \(z = 0\). For convenience, and without loss of generality, we will suppose that \(P(0) = 1\).

First, recall (cf. [1]) that the stability of the denominator depends on the stability of the term \(P(\exp(-sh/N))\) which means that the polynomial \(P(z)\) must have all its roots outside the unit disk. (We will suppose so, later on).

Now, remark that multiplying the expression of the denominator by \(\exp(sh/N)\) does not change the locus of the zeros. So, if we call \(P(z)\) the polynomial \(z^N P(1/z)\) (where \(N\) is the degree of \(P\), whose roots are the converses of those of \(P\), the stability of the denominator comes to the one of the following expression:

\[
(s - a) P(\exp(sh/N)) + \Delta \exp(a h).
\]

In this instance, let us denote by \(H(s)\) the first term \((s - a) P(\exp(sh/N))\), and by \(\Gamma\) the closed curve composed by the segment of the imaginary axis \(\Gamma_i = [-iR, iR]\) and the bow \(\Gamma_e = \{ R \exp(i\theta), \theta \in [-\pi/2, \pi/2]\} \).

From Cauchy’s theory (cf. [4]), we know that, since \(H(s)\) is holomorphic in the complex plane, the number of zeros of \(H(s) - z_o\) enclosed by \(\Gamma\) taken in the counterclockwise direction, is equal to

\[
\frac{1}{2i\pi} \int_{\Gamma} \frac{H(s)}{H(s) - z_o} ds.
\]

what can be written

\[
\frac{1}{2i\pi} \int_{H(\Gamma)} \frac{dz}{z - z_o} = \text{Ind}_{H(\Gamma)}(z_o),
\]

which is the number of turns of \(H(\Gamma)\) around the complex number \(z_o\).

Therefore, the stability of the denominator comes to the nullity of \(\text{Ind}_{H(\Gamma)}(z_o)\) for \(z_o = -\Delta \exp(a h)\).

That is why we will investigate the conditions of existence of reals \(z_o\) whose index according to the closed curve \(H(\Gamma)\) is null.

### 3 Study of \(H(\Gamma)\)

Here, we study the closed curve \(H(\Gamma)\) by the way of the evolution of the argument of \(H(s)\) when \(s\) traces out \(\Gamma\).

Let \(\beta_i, i = 1, ..., N\), be the roots of the polynomial \(P\), which are all inside the open unit disk.

First, consider the image by \(H\) of the bow \(\Gamma_e\). When \(s = Re^{i\theta}\), the expression of \(H(s)\) is

\[
(Re^{i\theta} - a) e^{R h \cos \theta} \prod_{i=1}^{N} (e^{i \frac{\beta_i h}{N} \sin \theta} - e\frac{\beta_i h}{N} \cos \theta),
\]

whose argument is the sum of the argument of \((Re^{i\theta} - a)\) and of those of each factor of the product. It is clear that the first one is increasing when \(\theta\) covers the interval \([-\pi/2, \pi/2]\), and the same conclusion holds rapidly for the other factors when \(R\) tends to infinity. Thus we know that \(H(\Gamma_e)\), after having taken some distance from the origin, turns around it in the counterclockwise direction.

Now, let

\[
H(\Gamma_i) = \{ iy - a, P(e^{iy}) \} , y \in [-R, R]\},
\]

that for convenience we will write

\[
\{ -a(1 - i \frac{\pi N}{ah}), P(e^{ix}) \} , x \in [-\frac{Rh}{N}, \frac{Rh}{N}]\}
\]

When \(x\) is increasing (what is not the case), the argument of the first factor:

\[
\theta_1(x) = \arg(1 - i \frac{\pi N}{ah})
\]

is decreasing whereas the argument of the value of the polynomial

\[
\theta_2(x) = \arg(P(e^{ix}))
\]
is increasing. Also, even if it is clear that the increasing part will soon prevail over the decreasing one, we need to compare their derivatives:

\[
\dot{\theta}_1(x) = -\frac{N}{ah} + \frac{1}{1 + \left(\frac{\Delta x}{ah}\right)^2},
\]

\[
\dot{\theta}_2(x) = \frac{1}{2} \frac{P'(e^{-ix})e^{-ix}}{P(e^{-ix})} + \frac{1}{2} \frac{P'(e^{ix})e^{ix}}{P(e^{ix})}.
\]

Indeed, if the argument of \(H(s)\) : \(\theta(x) = \theta_1(x) + \theta_2(x)\) is unceasingly decreasing when \(s = \frac{N}{ah}\) goes from \(iR\) to \(-iR\), that means that \(H(\Gamma_1)\) turns around the origin in the clockwise direction. After what \(H(\Gamma_\sigma)\) turns in the opposite direction around the origin but also around \(H(\Gamma_i)\), and since \(\text{Ind}_{H(\Gamma_1)}(0) = 1\) (as the number of zeros of \(H(s)\) enclosed by \(\Gamma\) once more than it). So, we can but conclude that, in this case, there does not exist any point \(\alpha\) in the complex plane for which \(\text{Ind}_{H(\Gamma_\sigma)}(\alpha) = 0\).

So, let us consider the inequality condition

\[
\dot{\theta}_1(x) + \dot{\theta}_2(x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{8}
\]

Since \(\dot{\theta}_2(x) > N/2, \forall x \in \mathbb{R}\), a sufficient condition that the argument \(\theta(x)\) is increasing with \(x\) is:

\[
-\frac{N}{ah} \frac{1}{1 + \left(\frac{\Delta x}{ah}\right)^2} + \frac{N}{2} \geq 0.
\]

What happens when \(x^2 \geq (2 - ah)\frac{ah}{N^2}\). Thus, as we could guess, the argument \(\theta(x)\) is unceasingly increasing with \(x\) out of a small interval which is empty when \(ah \geq 2\). A result we can state through the

**Proposition** Let \(\dot{x}(t) = ax(t) + u(t - h)\), a time-delay system with \(a \in \mathbb{R}^+, h \in \mathbb{R}^+\). If the product \(ah\) is greater or equal to 2, there does not exist a control law of the type

\[
u(t) = f(x(t)) + \sum_{n=0}^{N} \alpha_n u(t - \frac{n}{N}h),
\]

such that the closed-loop system is stable.

Conversely, if, for \(x = 0\), \(\dot{\theta}_1(x) + \dot{\theta}_2(x) < 0\), what can be written

\[
\frac{N}{ah} + \frac{P'(1)}{P(1)} < 0, \tag{9}
\]

that means there exists a loop enclosing in the clockwise direction an open set in the left half-plane and close to the origin. So, if \(H(\Gamma_i)\) does not come to cross the zone between the loop and the origin, what can be state saying that

\[
(1 - i\frac{N}{ah})\frac{P(e^{ix})}{P(e^{-ix})} > 1, \tag{10}
\]

when it belongs to \(\mathbb{R}^+\), we can conclude to the existence of points \(\alpha\) enclosed by the loop for which the index \(\text{Ind}_{H(\Gamma_\sigma)}(\alpha) = 0\).

### 4 Conclusion

Thus, we get two conditions for stabilizability. One necessary : \(ah < 2\); and one sufficient gathering the inequalities (9) and (10). One can hope that this last is also a necessary condition, although we did not get the proof.

Besides, and relatively to the inequality (10), which does not lead here to a parametrization of all the polynomials satisfying it, we can nevertheless give a class of such polynomials : those having all their roots inside the open segment \([0, 1]\).

At last, and it was the first motivation of this work, these results should help to find a “best approximation” of the control law (2) proposed by A. Manitius and A. Olbrot in [2], and lead to a practical implementation.

### References


