On a problem of robust control of parabolic system

Vyacheslav Maksimov
Institute of Mathematics and Mechanics
Ural Branch, Russian Academy of Sciences
16, S. Kovalevskoi str.
620219 Ekaterinburg, Russia
maksimov@imm.uran.ru

Keywords: Distributed parameter systems, robust control.

Abstract

A problem of robust control is discussed. This problem consists in attainment of a given goal set by a phase trajectory of a system describing the process of solidification of a liquid. It is assumed that an uncontrollable disturbance as well as a control act upon the system. An algorithm which is stable with respect to informational noises and computational errors is designed in the case of incomplete information on phase trajectory (only a "part" of coordinates may be measured). The work is based on investigations [1–4] and continues [5–11].

1 Statement of the problem

Consider the following system of equations in a domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \), with a sufficiently smooth boundary \( \Gamma \) (for example, from class \( C^2 \))

\[
\begin{align*}
\frac{\partial \psi(t, \eta)}{\partial t} + l \frac{\partial \varphi(t, \eta)}{\partial t} &= \triangle \psi(t, \eta) + u(t, \eta) - v(t, \eta), \\
\frac{\partial \varphi(t, \eta)}{\partial t} &= \triangle \varphi(t, \eta) + a(\eta)\varphi(t, \eta) + b(\eta)\varphi^2(t, \eta) - \varphi^3(t, \eta) + \psi(t, \eta), \\
(t, \eta) &\in Q = T \times \Omega, \quad T = [0, T]
\end{align*}
\]

(1)

with boundary

\[
\psi(t, \eta) = \phi(t, \eta) = 0, \quad (t, \eta) \in \Gamma \times T
\]

(2)

and initial

\[
\psi(0, \eta) = \psi_0(\eta), \quad \phi(0, \eta) = \phi_0(\eta), \quad \eta \in \Omega
\]

(3)

conditions. Here \( \triangle \) is the Laplace operator, \( \psi(t, \eta) \) describes the temperature of a medium, \( \phi(t, \eta) \) characterizes the distinction between solid and liquid subdomains of \( \Omega \), \( l > 0 \) is a constant, \( a(\eta), b(\eta) \in C(\overline{\Omega}) \) are given functions, \( \psi_0(\eta) \) and \( \varphi_0(\eta) \) (initial states of the system) are elements of space \( H^1(\Omega) \).

The system (1) simulates the process of solidification of a liquid. This system was under investigation in papers [12–14] (see also references in [14]), where theorems of the existence and uniqueness of classic and generalized solutions were proved, sufficient conditions of their regularity were derived, numerical solving methods were considered, the problem of dynamical reconstruction of an input was investigated and so on. At the same time, as it is noticed in [14], the questions of control of such system have received little attention. Only the problem of optimal control was investigated and necessary conditions of optimality in the form of principle of maximum [13, 14] were set. Our aim is different from the ideas of papers [13, 14]. We indicate the procedure of positional control of the system (1) through results of inaccurate measurements of values \( \phi(t) \) under the action of an unknown time-varying disturbance \( v(t), t \in T \).

A solution of (1)–(3) is understood as a unique function

\[
x(\cdot) = x(\cdot; 0, x_0, u(\cdot), v(\cdot)) \in V_T = \{W_2(\cdot) \cap C(T; H^1_0(\Omega)) \times \{W_2(\cdot) \cap C(T; H^1_0(\Omega))\},
\]

satisfying (1). Here \( W_2(T) \) is the standard space:

\[
W_2(T) = \{y(\cdot) \in L_2(T) : y(\cdot), y(\cdot), y(\cdot) \in L_2(T), y(\cdot) \in L_2(T)\},
\]

derivatives are understood in the generalized sense.

Let a control \( u = u(t) \in P \) be formed synchro with a process and an unknown disturbance \( v = v(t) \in R \) act upon the system (1)–(3). Here \( P, R \subset H \) are bounded closed sets ("resources" for control and disturbance respectively). A uniform partition of the time interval \( T = \{\tau_i\}_{i=0}^n \) with step \( \delta \) is chosen. At time moments \( \tau_i \) the phase coordinate \( \varphi(\tau_i) \) may be inaccurately measured. Results of measurements (elements \( \xi^h_i \in H \)) satisfy the inequalities

\[
|\xi^h_i - \varphi(\tau_i)|_{H_n} \leq h, \quad n = 2,
\]

where \( h \in (0, 1) \) is a value of measurement accuracy, symbol \( | \cdot |_{H_n} \) denotes norm in space \( H_n, H_n = L_2(\Omega) \), if \( n = 2, \)
$H_n = L_0(\Omega)$, if $n = 3$. It is required to indicate such a rule of forming feedback control for the system (1)–(3)

$$u(t) = u_i(\xi^b_i) \in P, \quad t \in [\tau_i, \tau_{i+1}], \quad i \in [0 : m - 1],$$

that, whatever the unknown disturbance $v = v(t) \in R$ maybe, the phase state of the system

$$x(t) = x(t; 0, x_0, u(\cdot), v(\cdot))$$
at the moment $t = \vartheta$ belongs to a sufficiently small $\varepsilon$-neighborhood of a given set $M \subset H \times H$ (i.e. set $M^\ast$).

The player should choose this rule in order to provide the property of trajectory mentioned above under any possible realization of disturbance $v = v(t)$. Note that the nature of disturbance $v$ is indifferent for us.

This disturbance may be a program control or a positional feedback control. It is necessary only that two conditions are fulfilled: first, realization $v(t)$ should be measurable (by Lebesque) function on $T$, second, it should satisfy the following inclusion: $v(t) \in R$ for a.a. $t \in T$. In the present paper an algorithm for solving the problem under consideration is indicated. This algorithm is based on the method of dynamical inversion (the method of dynamical control approximation) developed in [8–11] as well as on the well-known in the theory of positional control method of stable tracks [1].

In connection with incomplete information (namely, with the theory of positional control method of stable tracks [1]), we introduce in the control loop an additional block of dynamical reconstruction (approximation) of unknown coordinate $\psi(t)$ ("identification" block). This block plays the role of provider of information on current phase state of the system. The information is fed onto the control block functioning according to the given feedback.

## 2 The solving algorithm

Let us indicate the algorithm for solving the problem formulated above. Let

$$X_T = \{x(\cdot; 0, x_0, u(\cdot), v(\cdot)) : u(\cdot) \in P(\cdot) \land v(\cdot) \in R(\cdot)\} \subset V_T^{(1)}$$

be the bundle of all motions of the system (1)–(3) from initial state $x_0$.

$$P(\cdot) = \{u(\cdot) \in L_2(T; H) : u(t) \in P \text{ for a.a. } t \in T\},$$

$$R(\cdot) = \{v(\cdot) \in L_2(T; H) : v(t) \in R \text{ for a.a. } t \in T\},$$

$\Phi_T$ and $\Psi_T$ be projections of $X_T$ onto spaces

$$W^{2,1}_1(Q) \cap C(T; H_0^1(\Omega))$$
of coordinates $\psi$ and $\varphi$, respectively.

Let the following condition be fulfilled.

### Condition 1

a) There exists a convex closed set $D \subset H$ such that

$$P = R + D = \{z = x + y : x \in R, y \in D\}.$$  

b) There exists a control $t \to u_*(t) \in D$ for a.a. $t \in T$, which provides attainment of the set $M$ at the moment $t = \vartheta$ by a phase trajectory $x(\cdot) = x(\cdot; 0, x_0, u_*(\cdot))$ of the system

$$\frac{\partial \psi(t, \eta)}{\partial t} + i \frac{\partial \varphi(t, \eta)}{\partial t} = \Delta \psi(t, \eta) + u_*(t, \eta),$$

$$\frac{\partial \varphi(t, \eta)}{\partial t} = \Delta \varphi(t, \eta) + a(\eta)\varphi(t, \eta) + b(\eta)\varphi^2(t, \eta) - \varphi^3(t, \eta) + \psi(t, \eta), \quad (t, \eta) \in Q$$

with the boundary and initial conditions (2), (3): $x_*(\vartheta) = x(\vartheta; 0, x_0, u_*(\cdot)) \in M$.

To solve the problem following the approach [8–11], we introduce an auxiliary system described by the equations

$$\frac{\partial w_1^h(t, \eta)}{\partial t} - \Delta w_1^h(t, \eta) = u^h(t, \eta) + u_*(t, \eta), \quad t \in \delta_i = [\tau_i, \tau_{i+1}],$$

$$\frac{\partial w_2^h(t, \eta)}{\partial t} - \Delta w_2^h(t, \eta) = f^h(t, \eta), \quad \eta \in \Omega, \quad i \in [0 : m_h - 1],$$

with boundary

$$w_1^h(t, \eta) = w_2^h(t, \eta) = 0, \quad (t, \eta) \in T \times \Gamma$$

and initial

$$w(0) = x_0 = \{\varphi_0, \psi_0\} \in H_0^1(\Omega) \times H_0^1(\Omega).$$

conditions. Here

$$f^h(t, \eta) = a(\eta)\xi_i(\eta) + b(\eta)\xi_i^2(\eta) - \xi_i^3(\eta) + p^h(t, \eta),$$

$$u^h(t, \eta) = v_i^h(\eta), \quad t \in \delta_i, \quad i \in [1 : m_h - 1],$$

$$v_0^h = 0, \quad \xi_i = \xi_i^{h(\tau_i)},$$

$$v_i^h = v_i^{h(\xi_{i-1}, \xi_i)} = l(\delta_i^h)(\xi_{i-1} - \xi_i).$$

Let operator $A_*$ be defined in the following way:

$$(A_*(y), t, \eta) = (\Delta y_1(t, \eta), \Delta y_2(t, \eta))$$

for a.a. $(t, \eta) \in Q$ and all

$$y = (y_1(t), y_2(t)) \in D(A_*) = \{y = (y_1, y_2) \in H^2(\Omega) \times H^2(\Omega) : y_1(\eta) = y_2(\eta) = 0 \text{ for a.a. } \eta \in \Gamma\},$$

$$W^{1,2}_1(T; X) = \{y(\cdot) \in L_2(T; X) : y_1(\cdot) \in L_2(T; X)\}. $$
A function
\[ w^h(.) = \{u^h_1(\cdot), u^h_2(\cdot)\} = (7) \]
\[ w^h(\cdot): 0, w(0), v^h(\cdot), p^h(\cdot), \xi^h(\cdot), \]
satisfying (6) and belonging to \( V_T \), where
\[ V_T = \{ y(\cdot) \in W^{1,2}(T; X) \cap C(T; H^2_0(\Omega) \times H^1_0(\Omega)), \]
\[ \alpha_s y \in L_2(T; X) \}

is called a solution of the system (6), (7) corresponding to a control \( v^h(\cdot) \in L_2(T; H) \) and a piecewise constant function \( \xi^h(\cdot) \).

Proceed to the description of the algorithm. Fix a function \( \alpha(h) : (0, 1) \to R^+ \) (a regularizer)

and a family of partitions \( \Delta_h, h \in (0, 1) \) of the interval \( T \) with the properties:

\[ \Delta_h = \{ \tau_i \}_{i=0}^m, \quad \tau_i = \tau_{i-1} + \delta, \quad \delta = \delta(h), \quad \alpha(h) \to 0, \delta(h) \to 0, \]
\[ h\delta^{-1}(h) \to 0, \quad (h + \delta(h))\alpha^{-1}(h) \to 0 \quad \text{as} \quad h \to 0. \]

Then we organize the process of synchronous feedback control of the model (6) and real system (1) in such a way that for sufficiently small \( h \) and \( \delta \) the motion of the system (1) belongs at moment \( \delta \) to sufficiently small neighborhood of the set \( M, \) i.e. \( M^\varepsilon, \) for all possible realizations \( \psi(\cdot) \in R(\cdot). \)

For this purpose we fix \( h, \alpha(h) \) and \( \Delta_h \) before beginning of the work of the algorithm. This work is decomposed into \( m-1, m = m_h \) identical steps. At the \( i-th \) step carried out during the time interval \( \delta_i = [\tau_i, \tau_{i+1}] \) the following actions are fulfilled. First, "identification" block calculates the value

\[ p^h_i = p^h_i(\xi_i, w^h_2(\tau_i)) = \arg \min \{l(\alpha, u, s_i) : u \in P_1\}, \quad (8) \]

where
\[ P_1 = \{ u \in H : \|u\|_{H} \leq d \}, \]
\( d \) is an arbitrary number which is greater than
\[ \sup_{t \in T} \{ |\psi(t; 0, x_0, u(\cdot), v(\cdot))|_{H} : u(\cdot) \in P(\cdot), v(\cdot) \in R(\cdot)\}, \]
\[ l(\alpha, u, s_i) = 2(s_i, u)_H + \alpha\|u^2\|_H^2, \quad s_i = w^h_2(\tau_i) - \xi_i, \]
symbol \( (\cdot, \cdot)_H \) denotes the scalar product in \( H. \) Then control block determines the element

\[ u_i = u_i(p^h_i, w^h_2(\tau_i)) = \arg \min \{(p^h_i - w^h_1(\tau_i), u)_H : u \in P\}. \]

After that during the interval \( \delta_i \) the constant control
\[ u(t) = u_i(p^h_i, w^h_1(\tau_i)), \quad (9) \]
is fed onto the input of the real system, and control
\[ p^h(t) = p^h_i(\xi_i, w^h_2(\tau_i)) \]
is fed onto the input of the model. As a result of the action of these two controls and some unknown disturbance \( v(t), \]
\( t \in \delta_i, \) the system (1)–(3) passes from state \( \{\psi(\tau_i), \varphi(\tau_i)\} \)
to state \( \{\psi(\tau_{i+1}), \varphi(\tau_{i+1})\} : \)
\[ \psi(\tau_{i+1}) = \psi(\tau_{i+1}; \tau_i, \varphi(\tau_i), \varphi(\tau_i), u_i, v(\cdot)), \]
\[ \varphi(\tau_{i+1}) = \varphi(\tau_{i+1}; \tau_i, \varphi(\tau_i), y(\tau_i), u_i, v(\cdot)), \]
and the model (6) passes from state \( \{w^h_1(\tau_i), w^h_2(\tau_i)\} \) to state \( \{w^h_1(\tau_{i+1}), w^h_2(\tau_{i+1})\} : \)
\[ w^h_1(\tau_{i+1}) = w^h_1(\tau_{i+1}; \tau_i, w^h_1(\tau_i), w^h_2(\tau_i), \varphi(\cdot), p^h_i, v^h_i), \]
\[ w^h_2(\tau_{i+1}) = w^h_2(\tau_{i+1}; \tau_i, w^h_1(\tau_i), w^h_2(\tau_i), \varphi(\cdot), p^h_i, v^h_i). \]

At the next \( (i+1)-th \) step the analogous actions are repeated. The procedure stops at moment \( t = \theta. \) The following theorem takes place:

**Theorem 1** For any \( \varepsilon > 0 \) one can indicate \( h_\varepsilon = h_\varepsilon(\varepsilon), \)
such that for \( h \in (0, h_\varepsilon) \) the inclusion
\[ x(\psi; 0, x_0, u(\cdot), v(\cdot)) \in M^\varepsilon, \]
is true, whatever a disturbance \( v(\cdot) \in R(\cdot) \) maybe.

Before proving the theorem, we give an auxiliary statement.

**Lemma 1** The following relation takes place:
\[ \sup_{t \in T} \int_\Omega \left\{ \left| \psi^2(t, \eta) + |\nabla \psi(t, \eta)|^2 + |\nabla \varphi(t, \eta)|^2 + \varphi^2(t, \eta) \right| \right\} d\eta + \int_0^T \int_\Omega \left\{ \psi^2(t, \eta) + \Delta \psi^2(t, \eta) + \varphi_t^2(t, \eta) + \varphi^2(t, \eta) \right\} d\eta \leq \]
\[ d_\varepsilon = C\mu(\varphi_0, \psi_0) \forall (\psi(\cdot), \varphi(\cdot)) \in X_T. \]

Here \( d(P) = \sup\{ |u|_{H} : u \in P \}, \quad d(R) = \sup\{ |u|_{H} : u \in R \}, \)
a constant \( C \) depends on \( |\varphi(\cdot)|_{H}, |\varphi(\cdot)|_{H}, \quad l, \mu(\varphi_0, \psi_0) = |\varphi_0|_{H} + |\psi_0|_{H} \}

In virtue of the Lions embedding theorem for \( n = 3 \) space
\( H^1(\Omega) \) \( (H^2(\Omega)) \) is embedded into \( L_2(\Omega) \) \( (L_\infty(\Omega)) \) compactly. Therefore the next statement follows from Lemma 1.

**Corollary 1** For \( n = 3 \) the set \( X_T \) is bounded in
\( L_2(T; L_\infty(\Omega) \times L_\infty(\Omega)) \cap C(T; L_0(\Omega) \times L_0(\Omega)). \)

Let \( \Xi(\varphi(\cdot), h) \) be the set of all "measurements" compatible with \( \varphi(\cdot) \in \Phi_T, \) i.e. \( \Xi(\varphi(\cdot), h) \) is the set of all piecewise constant functions \( \xi^h(\cdot), \xi^h(t) = \xi^h_i \) for \( t \in [\tau_i, \tau_{i+1}], i \in [0 : m_h - 1], \) with the properties (4).

**Lemma 2** Whatever \( h \in (0, 1), \xi^h(\cdot) \in \Xi(\varphi(\cdot), h), \psi(\cdot) = (\psi(\cdot), \varphi(\cdot)) \in X_T \) maybe, the following inequalities hold
\[ \|p^h(\cdot) - \psi(\cdot)\|_{L_2(T; H)}^2 \leq K\nu(h), \quad (10) \]
\[ |\varphi(t) - u^2(t)|^2_{H} + \int_0^t \int_\Omega \left\{ |\nabla (\varphi(t, \eta) - u^2(t, \eta))|^2 \right\} d\eta \, d\tau \leq \]
\[ K_0(h + \delta(h) + \alpha(h)). \]
Here constants $K$ and $K_0$ depend on $X_T$ and do not depend on $h, \xi^h(\cdot), x(\cdot)$:

$$\nu(h) = (h + \delta(h) + \alpha(h))^{1/2} + (h + \delta(h))^{-1/2}.$$  

**Proof.** Fix $h \in (0, 1)$, $\xi^h(\cdot) \in \Xi(\varphi(\cdot), h), x(\cdot) \in X_T$. From (1), (6) it follows in a usual way that

$$\frac{d\xi^h(t)}{dt} = (\mu^h(t), \mu^h(t)) + \int_{\Omega} [\nabla \mu^h(t, \eta)]^2 d\eta + 0.5 \alpha(h) \{p^h(t) - \psi(t)\}^2 \leq (E(t, \varphi(\cdot), \xi^h(\cdot)), \mu^h(t)) + h \{p^h(t) - \psi(t)\}^2$$  

for a. a. $t \in T$.

Multiply the second equation (6) by $w^h_2(t, \eta)$. Then we use the Green formula, Lemma 1 and inequality (4). Integrating by $t$, we obtain for $w^h_2(\cdot) = w^h_2(\cdot, 0, w(0), v^h(\cdot), p^h(\cdot), \xi^h(\cdot))$ the estimate

$$\sup_{t \in T} ||w^h_2(t)||_H^2 + \int_{\Omega} \int_{\Omega} [\nabla w^h_2(t, \eta)]^2 d\eta d\tau \leq C_1 \mu(\varphi_0, \psi_0)$$

which is uniform with respect to $h \in (0, 1)$, $\xi^h(\cdot) \in \Xi(\varphi(\cdot), h), x(\cdot) \in X_T, p^h(\cdot) \in P_1(\cdot) = \{u(\cdot) \in L^2(T; H) : u(t) \in P_1\}$ for a. a. $t \in T$. In the same manner, we multiply the second equation (6) by $w^h_2(t, \eta)$ and then by $\Delta w^h_2(t, \eta)$ and, after integration, we derive

$$\sup_{t \in T} \int_{\Omega} [\nabla w^h_2(t, \eta)]^2 d\tau + \int_{\Omega} \int_{\Omega} [\nabla w^h_2(t, \eta)]^2 d\eta d\tau \leq C_2 \mu(\varphi_0, \psi_0).$$

In virtue of Lemma 1 and inequality (12), the following inequalities are valid:

$$(p^h(t) - \psi(t), \mu^h(t)) \leq C_2 \mu(\varphi_0, \psi_0)$$  

$$(p^h(t) - \psi(t), s_i)_{H^2} + k_1 \psi(t, h) \quad \text{for a. a.} t \in \delta_1.$$

$$\left\| w^h_2(t) \right\|_{L^2(T; L^\infty(\Omega))} \leq C_3 \mu(\varphi_0, \psi_0),$$

$$\varphi(t) = h + \int_{\tau}^t \left\{ \left| w^h_2(\tau) \right|_{H^2} + \left| \varphi(\tau) \right| \right\} d\tau.$$  

Here and in (12) constants $C_1-C_3$ depend on $|\alpha|_{C(\Omega)}, |\beta|_{C(\Omega)}, \vartheta, \eta$. In addition, by the inclusion $\xi^h(\cdot) \in \Xi(\varphi(\cdot), h)$, boundedness of the set $\Xi = \cup \{\Xi(\varphi(\cdot), h) : h \in (0, 1), \xi(\cdot) \in \Phi_T\} \subset C(T; H_n(\Omega))$, corollary 1 and estimates (13) we deduce that

$$\left\| (E(t, \varphi(\cdot), \xi^h(\cdot)), \mu^h(t)) \right\| \leq \left\| (E(t, \varphi(\cdot), \xi^h(\cdot)), \mu^h(t)) \right\|_{H^2} \leq C_3 \mu(\varphi_0, \psi_0).$$

Note that in (14) $\mu^h(t)$ may be replaced by $\psi(t)$. Thus, from (11), (13), (14), we, taking into account the rule of determination of $p^h(\cdot)$ (see (8)), get

$$\varepsilon_h(t) = \varepsilon_h(t_i) + k_4 (t - \tau_i) \phi_i(t, h) \quad \text{for } t \in \delta_1.$$  

Therefore

$$\varepsilon_h(t) = 0.5 \left| \varphi(t) - w^h_2(t) \right|^2 + \int_0^t \int_{\Omega} \left| \nabla (\varphi(\tau, \eta) - w^h_2(\tau, \eta)) \right|^2 d\eta \, d\tau + 0.5 \alpha(h) \left\{ \left| p^h(\tau) \right|^2_{H^2} - \left| \psi(\tau) \right|^2_{H^2} \right\} d\tau \leq \varepsilon_h(0) + k_4 (h + \delta) \int_0^t \left\{ \left| w^h_2(\tau) \right|_{H^2} + \left| \varphi(\tau) \right| \right\} d\tau \leq k_5 (h + \delta) \quad \text{for } t \in T.$$  

Then we have

$$\left\{ \mu^h(t), \psi(t) \right\}_{H^2} + \int_{\Omega} \nabla \mu^h(t, \eta) \nabla \psi(t, \eta) \, d\eta = (E(t, \varphi(\cdot), \xi^h(\cdot)), \psi(t) \right\}_{H^2} + \left( (p^h(t) - \psi(t), \psi(t)) \right)_{H^2} \quad \text{for a. a.} t \in T.$$  

The following estimates follow from (14)–(16)

$$\left\| p^h(\cdot) \right\|_{L^2(T; H^2)}^2 \leq \left\| \psi(\cdot) \right\|_{L^2(T; H^2)}^2 + k_5 (h + \delta) \alpha^{-1},$$

$$\left| \int_0^t \int_{\Omega} \left( \left| p^h(t) - \psi(t) \right| - \psi(t) \right) \, d\tau \right| \leq k_6 (h + \delta^1/2) +$$
These estimates are uniform with respect to \( h \in (0,1), \xi h(\cdot) \in \Xi(\phi(\cdot), h), x(\cdot) = (\psi(\cdot), \phi(\cdot)) \in X_T \). Thus, in virtue of (17), (18), we obtain

\[
\|\psi(\cdot) - p^h(\cdot)\|_{L_2(T; H)} \leq 2 \|\psi(\cdot)\|_{L_2(T; H)} - 2\int_0^T (p^h(t), \psi(t))_H dt + k_5(h + \alpha)\alpha^{-1} \leq 2k_\tau(h + \alpha)^{1/2} + k_5(h + \alpha)\alpha^{-1} \leq K\nu(h).
\]

The lemma is proved.

**The proof of the theorem.** To prove the theorem, we estimate the value

\[
L_h(t) = |\psi(t) - w^h_2(t)|^2_H + 2|\phi_2(t) - w^h_2(t)|^2_H + \|\psi(t) - w^h_1(t)|^2_H + |\phi_1(t) - w^h_1(t)|^2_H,
\]

where

\[
x(t) = \{\psi(t), \phi(t)\} = \\
\{\psi(t; 0, x_0, u(\cdot), v(\cdot)), \phi(t; 0, x_0, u(\cdot), v(\cdot))\},
\]

\[
x_*(t) = \{\psi_*(\cdot), \phi_*(\cdot)\} = \\
\{\psi(t; 0, x_0, u_*(\cdot), v(t; 0, x_0, u_*(\cdot))\}
\]

are solutions of the systems (1) and (5), respectively. Introduce the notation

\[
|\phi(\cdot)|^2_{H_1} = \frac{1}{2} |\psi(\cdot)|^2_{H} + \int_0^T |\nabla \phi(\tau, \eta)|^2 d\eta d\tau,
\]

\[
\phi(\cdot) \in W^{2,1}(Q).
\]

Similar to (11) we obtain

\[
0.5\lambda_h^{(1)}(t) = (\psi(t) - w^h_1(t), \psi(t) - w^h_1(t))_H + \int_\Omega |\nabla (\psi(t, \eta) - w^h_1(t, \eta))|^2 d\eta \leq I_i(t)
\]

for a. a. \( t \in [\tau_i, \tau_{i+1}) \).

Here

\[
\lambda_h^{(1)}(t) = \|\psi(t) - w^h_1(t)|^2_{H_1},
\]

\[
I_i(t) = \left(1 - \delta^{-1}(h) - \delta^{-1} - \psi(t) + u(t) - v(t) - u_*(t),
\]

\[
\psi(t) - w^h_1(t))_H.
\]

Evaluate \( I_i \). Note that, in virtue of Condition 1, the following equality is true:

\[
u_0(t) = u_0(t) - v(t) \quad \text{for a. a.} \quad t \in T, \tag{20}
\]

\( u_0(\cdot) \) is some function from \( P(\cdot) \). Then, whatever the set \( S_\star \) bounded in space \( W^{1,2}(T; H) \) maybe, the inequality is true uniformly with respect to all \( \psi(\cdot) \in \Phi_T \):

\[
\sup_{\tau, \eta \in S_\star} \left| \int_0^\tau \left( t\varphi(\tau) + v^h(\tau))_H dt \right| \leq \tag{21}
\]

\[
k_0\{h\delta^{-1} + \delta^{1/2}\},
\]

where \( k_0 = k_0(S_\star), \delta = \delta(h), v^h(t) = v^h_1(t), t \in \delta_i, i \in \{0 : m_i - 1\} \). Multiplying the first equation (6) by \( w^h_1(t) = w^h_1(\cdot; 0, u(0), v^h_1(\cdot), \xi h(\cdot), p^h(\cdot)) \) and integrating, we derive similar to (12) the estimate

\[
\sup_{t \in T} |w^h_1(t)|_H + \int_0^\tau \left\{ |\nabla w^h_1(\tau, \eta)|^2 + \Delta^2 w^h_1(\tau, \eta) \right\} d\eta d\tau + \|w^h_1(\cdot)|_{L_2(T; H)} + \|w^h_1(\cdot)|_{L_2(T; L_0(\Omega)\cap C(T; H_1))} \leq \tag{22}
\]

\[
C_{4\mu}(\phi_0, \psi_0)
\]

which is uniform with respect to \( \xi h(\cdot) \in \Xi(\phi(\cdot), h), h \in (0,1), x(\cdot) \in X_T, p^h(\cdot) \in \Phi_1(\cdot) \). In virtue of (20)–(22), Lemma 1 and the rule of determination of \( u(t) \) (see (9)) we have for \( \nu \in \{\tau_j, \tau_{j+1}\}, j \in \{0 : m - 1\} \)

\[
\int_{\tau_j}^{\tau_{j+1}} I_j(t) dt + \sum_{i=0}^{j-1} \int_{\tau_i}^{\tau_{i+1}} I_i(t) dt \leq k_1 \{h\delta^{-1} + \delta^{1/2}\} \tag{23}
\]

\[
+ \sum_{i=0}^{j-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ (u(t) - u_0(t), p^h_1 - w^h_1(\tau))_H + 2d(P) |p^h_1 - \psi(t)|_H + k_2 \int_{\tau_i}^{\tau_{i+1}} |w^h_1(\tau)|_H d\tau \right\} dt + \int_{\tau_j}^{\tau_{j+1}} I_j(t) dt \leq \]

\[
k_1 \{h\delta^{-1} + \delta^{1/2}\} + k_3 \sum_{i=0}^{j-1} \int_{\tau_i}^{\tau_{i+1}} \left\{ \delta^{1/2} + |p^h_1 - \psi(t)|_H \right\} dt.
\]

In turn, from (19), (23) it follows that

\[
\lambda_h^{(1)}(t) \leq k_4 (\delta^{1/2} + h\delta^{-1} + \int_0^\tau |p^h(\tau) - \psi(\tau)|_H d\tau). \tag{24}
\]

Besides, taking into account Lemma 2, we conclude that the estimate

\[
\lambda_h^{(2)}(t) = |\phi(\cdot) - w^h_2(\cdot)|^2_{H_1} \leq \tag{25}
\]

\[
\delta^{-1}(h)\{h\delta^{-1} + \delta^{1/2}\} + k_5 \{h\delta^{-1} + \delta^{1/2}\}.
\]
Lemma 2. For

\[ \kappa_0 (h + \delta (h) + \alpha (h)) \quad \forall \ t \in T. \]

is true. Then, it is easily seen that

\[ \lambda_h^{(3)} (t) = |\psi_s (t) - w_h (t)|_H^2 \leq \]

\[ \int_0^t \int (v^h (\tau) + l \varphi (\tau), \psi_s (\tau) - w_h (\tau))_H \, d\tau. \]

By Corollary 1 and estimates (22)

\[ |(F_{\varphi_s} (\tau) - F_{\varphi} (\tau), \psi_s (\tau) - w_h (\tau))_H| \leq \]

\[ \nu (t; \varphi (\cdot), \varphi_s (\cdot)) |\varphi (t) - \varphi_s (t)|_H |\psi_s (t) - w_h (t)|_H, \]

\[ \sup_{\varphi (\cdot), \varphi_s (\cdot)} \int_0^t \nu^2 (t; \varphi (\cdot), \varphi_s (\cdot)) \, dt \leq c. \]

Using the Green formula, we deduce from (26) that

\[ l |(\varphi_s (\tau) - \varphi (\tau), \psi_s (\tau) - w_h (\tau))_H| \leq \]

\[ l^2 \int |\nabla (\varphi_s (\tau, \eta) - w_h (\tau, \eta))|^2 \, d\eta + \]

\[ l^2 \int |\nabla (\varphi_s (\tau, \eta) - w_h (\tau, \eta))|^2 \, d\eta + \]

\[ \frac{1}{2} |(\nabla (\psi_s (\tau, \eta) - w_h (\tau, \eta))|^2 \, d\eta + \]

\[ 2 \nu (t; \varphi (\cdot), \varphi_s (\cdot)) |\varphi (t) - w_h (\tau)|_H + \]

\[ |\psi_s (\tau) - w_h (\tau)|_H^2 + |\psi_s (\tau) - w_h (\tau)|_H^2 \}. \]

Here

\[ F_{\varphi_s} (\tau) = a \varphi (\tau) + b \varphi^2 (\tau) - \varphi^3 (\tau). \]

In addition, by (21), (22) the inequality

\[ \sup_{t \in T} \left| \int_0^t \int (v^h (\tau) + l \varphi (\tau), \psi_s (\tau) - w_h (\tau))_H \, d\tau \right| \leq \]

\[ k_8 \{ h \delta^{-1} + \delta^{1/2} \} \]

is true uniformly with respect to all \( w_h (\cdot) \). Thus, using again Lemma 2, for \( t \in T \) we obtain

\[ \lambda_h^{(3)} (t) \leq \int_0^t (l^2 \int |\nabla (\varphi_s (\tau, \eta) - w_h (\tau, \eta))|^2 \, d\eta + \]

\[ \frac{1}{2} |(\nabla (\psi_s (\tau, \eta) - w_h (\tau, \eta))|^2 \, d\eta + \]

\[ 2 \nu (t; \varphi (\cdot), \varphi_s (\cdot)) |\varphi (t) - w_h (\tau)|_H + \]

\[ |\psi_s (\tau) - w_h (\tau)|_H^2 \} \right) \, d\tau + \]

\[ k_0 \left( 1 + \int_0^t \nu (\tau; \varphi (\cdot), \varphi_s (\cdot)) \, d\tau \right) \{ h \delta^{-1} + \delta^{1/2} + \alpha (h) \}. \]

Then we have

\[ \lambda_h^{(4)} (t) = t^2 |e(t)|_H^2 \leq \]

\[ l^2 \int_0^t (F_{\varphi_s} (\tau) + \psi_s (\tau) - h (\tau, \eta), e(\tau))_H \, d\tau, \]

where \( e(t) = \varphi_s (t) - w_h (t) \). It is easy to show (see (14) and (26)), that for a. a. \( t \in T \)

\[ |(F_{\varphi_s} (\tau) + \psi_s (\tau) - h (\tau, \eta), e(\tau))_H| \leq \]

\[ |(F_{\varphi_s} (\tau) + \psi_s (\tau) - h (\tau, \eta), e(\tau))_H| + \]

\[ |(\psi_s (\tau) - p^h (\tau), e(\tau))_H| + k_7 (h + \delta^{1/2}) \leq \]

\[ \nu (t; \varphi (\cdot), \varphi_s (\cdot)) |\varphi (t) - \varphi_s (t)|_H |e(\tau)|_H + \]

\[ |(\psi_s (\tau) - p^h (\tau), e(\tau))_H| + k_7 (h + \delta^{1/2}). \]

In this case from (29) and (22) it follows that

\[ \lambda_h^{(4)} (t) \leq k_8 \int_0^t \left\{ (1 + \nu^2 (\tau; \varphi (\cdot), \varphi_s (\cdot))) |e(\tau)|_H^2 + \right\} \]

\[ |\psi (\tau) - w_h (\tau)|_H^2 + |\psi_s (\tau) - w_h (\tau)|_H^2 + \]

\[ + |\psi (\tau) - p^h (\tau)|_H \right\} d\tau + k_9 (h + \delta^{1/2} + \alpha (h)). \]

From (24), (25), (28), (30) we get for sufficiently small \( h, h \in (0, h_1) \):

\[ L_h (t) \leq k_{10} \left\{ h \delta^{-1} + \delta^{1/2} + \alpha + \right\} \]

\[ \int_0^t (1 + \nu^2 (\tau; \varphi (\cdot), \varphi_s (\cdot))) L_h (\tau) \, d\tau + \]

\[ \int_0^t |p^h (\tau) - \psi (\tau)|_H \, d\tau \}, \quad t \in T. \]

At last, by the Gronwall inequality and Lemma 2 we conclude that

\[ L_h (t) \leq k_{11} \left\{ h \delta^{-1} (h) + (h + \delta (h) + \alpha (h))^{1/4} + \right\} \]

\[ (h^{1/2} + \delta^{1/2} (h)) \alpha^{-1/2} (h), \quad t \in T. \]

Taking into account the properties of functions \( \alpha (h) \) and \( \delta (h) \) (see (7)), we deduce that one can determine for any \( \varepsilon > 0 \) such number \( h_\varepsilon > 0 \) (in an explicit form) that for \( h \in (0, h_\varepsilon) \) the inclusion \( \{ \varphi (\vartheta), \psi (\vartheta) \} \in M^\varepsilon \) is fulfilled. The theorem is proved.
Acknowledgments

This work was supported in part by the Russian Foundation for Basic Research (project 98-01-00046), INTAS (project 96-0816) and ISTC (project 99-1293).

References


