Abstract

This paper discusses the fault-tolerance of symmetric systems by revealing the underlying mathematical mechanism of the loss of controllability for symmetric systems induced by failures. Based on the decomposition of the symmetric systems into subsystems under the symmetry, the controllability of the entire system can be discussed by checking that of each subsystem. The analysis of the fault-tolerance in this paper is an extension of this idea with the aid of the chain-adapted transformation matrix for the decomposition.

1 Introduction

Mathematical treatment of the symmetry found in various phenomena is systematized based on the group theory, for instance, in bifurcation theory [2], quantum mechanics [11], structural engineering [5] and so on. Also in the field of control theory, scattered researches have been carried out concerning the control of group-theoretic symmetric systems [3][7][10]. In [3], it is shown that the characteristics of the symmetric systems can be investigated by those of subsystems obtained by the decomposition based on the group-theoretic symmetry.

For control systems in general, practical importance of the fault-tolerance has been fully recognized. That is, the systems are desired to retain some characteristics in spite of some failures. As for the fault-tolerance of symmetric control systems, however, there are only studies on the graph-theoretic connectivity [1][4] and few researches on the control theoretic characteristics. This paper will discuss the fault-tolerance of symmetric systems with respect to controllability, which is a fundamental characteristic of control systems. We consider the controllability of a system as a characteristic that should be retained in spite of failures in some control channels, and clarify those failures which cause the symmetric system to lose its controllability.

A first attempt in this direction is found in [9], where an interesting relationship between the symmetry and the fault-tolerance has been observed. That is, when some failures cause a symmetric system to be uncontrollable, the system after the failures has certain symmetry as well. For example, consider a $D_9$-symmetric system that consists of nine identical modules connected in a ring (Fig. 1(a)). The arrows in the figure represent effective inputs. Then, the failures shown in Fig. 1(b) turn out to cause the system to be uncontrollable. As can be seen from the figure, the system after the failures retains a partial ($D_3$-) symmetry. Conversely, if the system after failure is completely nonsymmetric, the entire system keeps its controllability (Fig. 1(c)). From the observation above, we can deduce that symmetric failure patterns tend to cause the symmetric systems to lose their controllability. Now, a question comes about if all the symmetric failures cause general symmetric systems to be uncontrollable or not.

The underlying mathematical mechanism of such results will be revealed in this paper. Whereas [9] has dealt with the restricted class of symmetric systems whose symmetry as a whole originates in the identity of the modules and certain regularity of their connections, the present paper will be concerned with systems with more general symmetry, and will show the mechanism of the loss of controllability induced by failures. The example shown in Fig. 2 illustrates our main result in this paper: Consider a $D_6$-symmetric spherical diamond system (Fig. 2(a)). According to our result, the system shown in Fig. 2(b) turns out to be uncontrollable because of the $D_3$-symmetric failures, whereas the system shown in Fig. 2(c) turns out to retain its controllability despite the $D_3$-symmetry in the failures.

The controllability of the entire system can be discussed by checking that of each subsystem, based on the decomposition [3] of the symmetric systems into subsystems under the group-theoretic symmetry. The analysis of the fault-tolerance in this paper is an extension of this idea with the aid of the chain-adapted transformation matrix for the decomposition.

The outline of this paper is as follows. In Section 2, we show the main results that reveal the mechanism of the loss of controllability for systems with general symmetry. There, the standard results in the group representation theory are
2 Group Theoretic Treatment of Failure in Symmetric Systems

2.1 Symmetric failures in symmetric systems

As explained in Introduction, the main concern of this paper is to reveal the mathematical mechanism of the loss of controllability for symmetric systems caused by symmetric failures. In this subsection we will formulate the notions of controllability for symmetric systems caused by symmetric failures. Finally, concluding remarks are given in Section 5. Preliminaries from group representation theory are given in Appendix.

Consider a linear time-invariant system $S$ that consists of $m$ control modules $\{S_1, S_2, \ldots, S_m\}$, each of which has its own control channel. The entire system $S$ is then described by

$$\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m
\end{bmatrix} = \begin{bmatrix}
  A_{11} & A_{12} & \cdots & A_{1m} \\
  A_{21} & A_{22} & \cdots & A_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  A_{m1} & A_{m2} & \cdots & A_{mm}
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m
\end{bmatrix} + \begin{bmatrix}
  B_{11} & B_{12} & \cdots & B_{1m} \\
  B_{21} & B_{22} & \cdots & B_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  B_{m1} & B_{m2} & \cdots & B_{mm}
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_m
\end{bmatrix},$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{r_i}$ denote the state of $S_i$ and the input from its control channel, respectively, with $\mathbb{R}^n$ being the set of real vectors of the size $n$. The diagonal elements $A_{ii}$ and $B_{ii}$ in (1) represent the effects of each module $S_i$ on its own state $x_i$. The off-diagonal elements $A_{ij}$ and $B_{ij}$ ($i \neq j$) imply the interactions among modules through the interconnections. Therefore, if two modules $S_i$ and $S_j$ ($i \neq j$) are not connected, the corresponding matrices $A_{ij}$ and $B_{ij}$ are zero. By denoting the state and the input of the entire system $S$ as

$$\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m
\end{bmatrix} \in \mathbb{R}^n, \quad \begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_m
\end{bmatrix} \in \mathbb{R}^r,$$

respectively, the equation (1) can be given in the standard form of a state transition equation:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Among the systems described by (3), we are interested in ones with group-theoretic symmetry. We say that the system (3) is symmetric with respect to a finite group $G$ if

$$T(g)A = AT(g), \quad T(g)B = BS(g), \quad g \in G,$$

where $T$ and $S$ are unitary representations of $G$ on $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively (see Appendix for terminology from group representation theory). The equation (4) often reflects the underlying geometric symmetry in the system structures. It should be mentioned here that we are interested in the characteristics of a system determined by its symmetric structure and not by the numerical information of the system matrices $A$ and $B$.

In order to discuss the fault-tolerance of the symmetric systems, we restrict the failure here to that of the control channels. Generally, if a control channel of the module, say $S_i$, is in the outage or replacement, the control input $u_i(t)$ has no correlation with the state $x(t)$. The input from the control channel in the outage has no influence on the state of the entire system. This situation is described in the mathematical model (1) by

$$u_i(t) = 0 \quad (S_i \text{ is in the outage}),$$

that is, $S_i$ is supposed to have no control input from its own control channel.

According to the failure defined in (5), let $M$ and $N$ denote the index sets of the functioning modules and of the modules in the outage, respectively. The failure pattern of the system is thus described by the pair of $M$ and $N$. In addition, we
introduce the failure matrix $F$ of order $r$ in such a way that the matrix $BF$ has zero column blocks that correspond to the control channels in the outage. Such a matrix $F$ is given by $F = \bigoplus_{i=1}^m F_i$ with

$$F_i = \begin{cases} I_{r_i}, & (i \in M), \\ O_{r_i}, & (i \in N), \end{cases}$$

where the matrices $I_k$ and $O_k$ denote, respectively, the unit matrix and the zero matrix of order $k$ in general. This means

$$F = F_M \oplus F_N = I_f \oplus O_{r-f}$$

(6)

with $F_M = \bigoplus_{i \in M} F_i = I_f$, $F_N = \bigoplus_{i \in N} F_i = O_{r-f}$, $f = \sum_{i \in M} r_i$, by an appropriate permutation of the indices of the modules. Note that any failure pattern can be given in the form of (6) and that the system after the failures is denoted as $(A, BF)$.

The symmetry of a failure pattern can then be formulated similarly to (4) for the matrix $F$ in (6). A failure pattern $F$ is said to be symmetric with respect to a subgroup $H$ of $G$ if

$$S(h)F = FS(h), \quad h \in H.$$  

(7)

Note that the symmetry of failure patterns is defined with respect to a subgroup $H$ of $G$. Given a subgroup $H$ of $G$, we have a set of failure patterns $F$ that satisfy (7). Conversely, for a given $F$,

$$H(F) = \{ g \in G \mid S(g)F = FS(g) \}$$

is a subgroup of $G$ and can be chosen as the subgroup $H$ in (7).

From the assumption above, the form of the unitary representation matrices $S(h)$ satisfying (7) is restricted to

$$S(h) = S_M(h) \oplus S_N(h), \quad h \in H,$$  

(8)

where $S_M(h)$ is of order $f$ and $S_N(h)$ is of order $(r - f)$, corresponding to the blocks of $F$ in (6). Namely, the representation matrices $S(h)$ in (7) splits into diagonal blocks for all $h \in H$.

**Remark 1** From the equations (4) and (7), the system after the failures $(A, BF)$ with $B_F = BF$ is also symmetric with respect to $H$ in the sense of (4), since

$$T(h)A = AT(h), \quad T(h)B_F = B_FS(h), \quad h \in H.$$  

Therefore, the symmetry of the original system and that of the failure patterns yield a partial symmetry of the system after the failures. \hfill \Box

2.2 Main Results

The main result of this paper is stated in this subsection in the form of Theorem 1, which gives a group-theoretic condition for the controllability after an $H$-symmetric failure in a $G$-symmetric system.

In the following, we consider the state space $\mathcal{X} \simeq \mathbb{C}^n$ and the input space $\mathcal{U} \simeq \mathbb{C}^r$ for the simplicity of mathematical treatment although the systems formulated above are described on real spaces. Namely, we consider the complexifications of $\mathcal{X}$ and $\mathcal{U}$. The family of all nonequivalent irreducible matrix representations of $G$ is denoted by $\{ D_G^\mu \mid \mu \in \mathcal{R}(G) \}$, where $D_G^\mu$ is a unitary irreducible matrix representation, of dimension $N^\mu$, on $\mathbb{C}$ and $\mathcal{R}(G)$ is the index set for the complex irreducible representations of $G$. Let the representations $T(g)$ and $S(g)$ of $G$ be decomposed into diagonal blocks of irreducible representations (see Appendix or the next subsection) as

$$T = \sum_{\mu \in \mathcal{R}(G)} a^\mu \mu, \quad S = \sum_{\mu \in \mathcal{R}(G)} b^\mu \mu,$$  

(9)

where the nonnegative integers $a^\mu$ and $b^\mu$ are called the multiplicities of $\mu$ in $T$ and $S$, respectively.

Consider an $H$-symmetric failure $F$, where $H$ is a subgroup of $G$. The family of all nonequivalent irreducible matrix representations of $H$ is denoted by $\{ D_H^\nu \mid \nu \in \mathcal{R}(H) \}$, where the dimension of $D_H^\nu$ is denoted by $N^\nu$. Let the representations $S_M(h)$ and $S_N(h)$ of $H$, defined in (8), be decomposed into diagonal blocks of irreducible representations as

$$S_M = \sum_{\nu \in \mathcal{R}(H)} b_M^\nu \nu, \quad S_N = \sum_{\nu \in \mathcal{R}(H)} b_N^\nu \nu$$  

(10)

with the multiplicities $b_M^\nu$ and $b_N^\nu$ of $\nu$ in $S_M$ and $S_N$, respectively.

An important technical ingredient in our argument is the use of chain-adapted bases with respect to $G$ and its subgroup $H$. Let the restriction of irreducible representations $\mu$ of $\mathcal{R}(G)$ to $H$ be described as

$$\mu \downarrow H = \sum_{\nu \in \mathcal{R}(H)} \alpha^\nu \mu \nu, \quad \mu \in \mathcal{R}(G),$$  

(11)

with nonnegative integers $\alpha^\nu \mu$ representing the multiplicity of $\nu$ in $\mu \downarrow H$, the restriction of $\mu$ to $H$.

Then, a necessary condition of group-theoretic nature for the controllability is obtained as follows. The proof will be given in the next subsection.

**Theorem 1** A $G$-symmetric system $(A, B)$ retains its controllability in spite of an $H$-symmetric failure $F$ only if there exists no pair of irreducible representation $\mu$ of $G$ and $\nu$ of $H$ such that

$$a^\mu \neq 0, \quad \alpha^\nu \mu \neq 0, \quad b_M^\nu = 0,$$  

(12)

where $a^\mu, \alpha^\nu \mu$ and $b_M^\nu$ are defined by (9), (11) and (10), respectively. \hfill \Box

The three conditions in (12) are concerned with $T$, the pair of $G$ and $H$, and $S$, respectively. The condition (12) often
turns out to be sufficient, as we will see later in Section 4. Therefore, as a rule of thumb, we may hopefully expect that the system retains its controllability if the condition (12) is not satisfied by any \( \nu \in R(H) \).

Moreover, the following theorem is derived concerning the rank deficiency of the controllability matrix by \( H \)-symmetric failures.

**Theorem 2** By an \( H \)-symmetric failure in a \( G \)-symmetric system, the rank of the controllability matrix is reduced at least by
\[
\sum_{(\mu, \nu) \in \mathbf{F}} a_{\mu} \alpha_{\mu}^* N_{\nu},
\]
where \( \mathbf{F} \) denotes the family of all pairs \( (\mu, \nu) \) which satisfy (12), i.e.,
\[
\mathbf{F} = \{(\mu, \nu) \in R(G) \times R(H) \mid a_{\mu} \neq 0, \lambda_{\mu} \neq 0, b_{\mu} = 0\}.
\] (13)

Theorem 2 implies that a \( G \)-symmetric system \((A, B)\) becomes uncontrollable by an \( H \)-symmetric failure \( F \) if the subset \( \mathbf{F} \) is nonempty, which is the statement of Theorem 1.

### 2.3 Proofs

As promised in the previous subsection, we derive Theorems 1 and 2 in the following.

The spaces \( \mathcal{X} \) and \( \mathcal{U} \) are decomposed as direct sums of the invariant subspaces corresponding to \( D_{\mathcal{G}}^\mu \), which is (often) abbreviated to \( D^\mu \), as
\[
\mathcal{X} = \bigoplus_{\mu \in R(G)} a_{\mu} \mathcal{X}_{\mu}, \quad \mathcal{U} = \bigoplus_{\mu \in R(G)} b_{\mu} \mathcal{U}_{\mu},
\]
where the nonnegative integers \( a_{\mu} \) and \( b_{\mu} \) are called the multiplicities. We define two unitary matrices \( Z \) of order \( n \) and \( W \) of order \( r \) as
\[
Z = (Z_{\mu} \mid \mu \in R(G)), \quad W = (W_{\mu} \mid \mu \in R(G)),
\] (14)
where \( Z_{\mu} \in \mathbb{C}^{n \times a_{\mu} N_{\nu}} \) and \( W_{\mu} \in \mathbb{C}^{r \times b_{\mu} N_{\nu}} \) are defined as
\[
Z_{\mu} = (Z_{\mu}^i \mid 1 \leq i \leq a_{\mu}), \quad W_{\mu} = (W_{\mu}^j \mid 1 \leq j \leq b_{\mu}).
\] (15)
with \( Z_{\mu}^i \in C^{\mathcal{X}_{\mu} \times N_{\nu}} \) and \( W_{\mu}^j \in C^{\mathcal{U}_{\mu} \times N_{\nu}} \), being sets of bases of \( \mathcal{X}_{\mu} \) and \( \mathcal{U}_{\mu} \), respectively. Note that the unitarity of \( T \) and \( S \) allows us to choose the matrices \( Z \) and \( W \) to be unitary over \( \mathbb{C} \), i.e., \( Z^* Z = I_n \) and \( W^* W = I_r \), where \( Z^* \) and \( W^* \) are the transposed conjugate matrices of \( Z \) and \( W \), respectively. Since \( Z_{\mu}^i \) and \( W_{\mu}^j \) are bases of \( \mathcal{X}_{\mu} \) and \( \mathcal{U}_{\mu} \), respectively,
\[
T(g) Z_{\mu}^i = Z_{\mu}^i D_{\mathcal{G}}^\mu(g), \quad S(g) W_{\mu}^j = W_{\mu}^j D_{\mathcal{G}}^\mu(g), \quad g \in G,
\]
holds and thus
\[
T(g) Z^\mu = Z^\mu (\bigoplus_{i=1}^{a_{\mu}} D_{\mathcal{G}}^\mu(g)), \quad S(g) W_{\mu}^j = W_{\mu}^j (\bigoplus_{j=1}^{b_{\mu}} D_{\mathcal{G}}^\mu(g)).
\] (16)
where \( \bigoplus_{i=1}^{a_{\mu}} D_{\mathcal{G}}^\mu(g), g \in G \) denotes a block-diagonal matrix consisting of \( a_{\mu} \) identical diagonal blocks \( D_{\mathcal{G}}^\mu(g) \). Then the representations \( T(g) \) and \( S(g) \) of \( G \) are decomposed by \( Z \) and \( W \) into diagonal blocks of irreducible representations as
\[
Z^* T(g) Z = \bigoplus_{\mu \in R(G)} a_{\mu} \bigoplus_{i=1}^{N_{\nu}} D_{\mathcal{G}}^\mu(g), \quad W^* S(g) W = \bigoplus_{\mu \in R(G)} b_{\mu} \bigoplus_{j=1}^{N_{\nu}} D_{\mathcal{G}}^\mu(g).
\] (17)

From (4) and (17), the system matrices \( A \) and \( B \) are also block diagonalized by the same matrices \( Z \) and \( W \), from Schur’s lemma (see Appendix for details), as
\[
\tilde{A} = Z^* A Z = \bigoplus_{\mu \in R(G)} a_{\mu} \bigoplus_{i=1}^{N_{\nu}} A^\mu_i, \quad \tilde{B} = Z^* B W = \bigoplus_{\mu \in R(G)} b_{\mu} \bigoplus_{j=1}^{N_{\nu}} B^\mu_j,
\] (18)
with matrices \( A_{\mu} \in \mathbb{C}^{a_{\mu} \times a_{\mu}} \) and \( B_{\mu} \in \mathbb{C}^{b_{\mu} \times b_{\mu}} \). Note that \( \bigoplus_{\mu \in R(G)} A^\mu \) in (18) denotes a block-diagonal matrix consisting of \( N_{\nu} \) identical diagonal blocks \( A^\mu \).

We can take the matrix \( W_{\mu}^j \) (\( 1 \leq j \leq b_{\mu} \)) in (15) compatibly with (11) so that it is further decomposed as
\[
W_{\mu}^j = (W_{\mu}^{j, \nu} \mid \nu \in R(H)),
\] (19)
with each \( W_{\mu}^{j, \nu} \in \mathbb{C}^{r \times a_{\mu} N_{\nu}} \) being a set of bases of the invariant subspace corresponding to \( \nu \in R(H) \). Such basis
\[
W = (W_{\mu}^{j, \nu} \mid \mu \in R(G), 1 \leq j \leq b_{\mu}, \nu \in R(H)),
\]
said to be chain-adapted with respect to \( G \) and its subgroup \( H \), which is a key technical ingredient of the proof. Consequently, with the chain-adapted bases \( W_{\mu}^{j, \nu} \) in (19), the irreducible matrix representation \( D_{\mathcal{G}}^\mu(g) \) in the second expression of (17) is decomposed as
\[
D_{\mathcal{G}}^\mu(h) = \bigoplus_{\nu \in R(H)} \bigoplus_{i=1}^{a_{\mu}} D_{\mathcal{H}}^\nu(h), \quad h \in H,
\] (20)
that is,
\[
W^* S(h) W = \bigoplus_{\nu \in R(H)} \bigoplus_{i=1}^{a_{\mu}} \bigoplus_{j=1}^{b_{\mu}} D_{\mathcal{H}}^\nu(h), \quad h \in H.
\] (21)

Since the rank of the controllability matrix is invariant under state transformations,
\[
\text{rank } C(A, B_F) = \text{rank } C(Z^* A Z, Z^* B_F)
\] (22)
holds. From (18) together with the unitarity of \( W \) leading to \( Z^* B_F = Z^* B W \cdot W^* F \), the rank (22) is calculated as
\[
\text{rank } C(Z^* A Z, Z^* B_F) = \text{rank } \{ \bigoplus_{i=1}^{n} W^* F \}.
\]
Therefore, according to the decomposition (18), the controllability of the \( G \)-symmetric system \((A, B)\) after the \( H \)-symmetric failure \( F \) is equivalent to that of the system
\[
(\tilde{A}, \tilde{B}, W^* F) = \left( \bigoplus_{\mu \in R(G)} a_{\mu} A^\mu, \bigoplus_{\nu \in R(H)} B^\nu, \left( \bigoplus_{\mu \in R(G)} b_{\mu} D_{\mathcal{G}}^\mu(g) W^* F \right) \right),
\] (23)
The matrix \( (W^* F)^* F \) in (23) is denoted as
\[
(W^* F)^* F = \left[ \left( \bigoplus_{\mu \in R(G)} b_{\mu} D_{\mathcal{G}}^\mu(g) W^* F \right) \right] \left( \bigoplus_{\nu \in R(H)} \bigoplus_{i=1}^{a_{\mu}} \bigoplus_{j=1}^{b_{\mu}} D_{\mathcal{H}}^\nu(h) \right)
\]
with the failure matrix \( F \) in the form of (6), where \( W_{\mu}^N \) denotes the first \( f \) rows of \( W_{\mu} \) as
\[
W_{\mu} = \begin{pmatrix} W_{\mu}^N \\ W_{\mu}^{f+1} & \cdots & W_{\mu}^{N} \end{pmatrix}.
\] (24)
From (15) and (19), $W^\mu_M$ and $W^\mu_N$ in (24) are described as
\[
W^\mu_M = (W_{j\mu}^\mu | 1 \leq j \leq b^\mu, \nu \in R(H)),
\]
\[
W^\mu_N = (W_{j\mu}^\mu | 1 \leq j \leq b^\mu, \nu \in R(H)),
\]
where $W_{j\mu}^\mu \in C^f \times \alpha^\mu_\nu N^\nu$ and $W_{j\mu}^\mu \in C^{(r-f) \times \alpha^\mu_\nu N^\nu}$. Moreover, with reference to
\[
W_{\mu\nu}^{\mu\nu} = (W_{j\mu}^{\mu\nu} | 1 \leq j \leq b^\mu, \nu \in (\nu-f) \times \alpha^\mu_\nu N^\nu),
\]
the matrix $W_{\mu\nu}^{\mu\nu}$ is described as
\[
W_{\mu\nu}^{\mu\nu} = (W_{j\mu}^{\mu\nu} | \nu \in R(H)).
\] (25)

Consequently, if $(W^\mu_M)^*$ has a zero row block compatible with the block structure of $\bar{B}$, the controllability matrix is not of row-full rank, and hence the system becomes uncontrollable. Namely, we obtain the following lemma.

**Lemma 1** A $G$-symmetric system $(A, B)$ becomes uncontrollable by an $H$-symmetric failure $F$ if $^1 W^\mu_M = O$ for some $\mu \in R(G), \nu \in R(H)$ with $a^\mu_\mu \alpha^\mu_\nu \neq 0$.

(Proof) If $W^\mu_M = O$ for a pair of $\mu \in R(G)$ and $\nu \in R(H)$ with $a^\mu_\mu \alpha^\mu_\nu \neq 0$, the product $\bigotimes_{\nu = 1}^{N^\nu} B^\nu_{\nu}\bigotimes_{\nu = 1}^{N^\nu} F$ in (23) has a zero row block compatible with the decomposition
\[
\bigotimes_{\nu = 1}^{N^\nu} B^\nu_{\nu} = \bigotimes_{\nu \in R(H)} \bigotimes_{\nu \in R(H)} B^\nu_{\nu},
\]
corresponding to $N^\mu = \sum_{\nu \in R(H)} \alpha^\mu_\nu N^\nu$.

We are now to clarify a group-theoretic mechanism that produces a zero row block in $(W^\mu_M)^*$, that is, $W^\mu_M = O$.

**Lemma 2** $W^\mu_M = O$ if $b^\mu_\nu = 0$.

(Proof) The matrix $(W^\mu)^*F$ satisfies
\[
(W^\mu)^* F = (W^\mu)^* S(h)^* S(h) F
\]
\[
= \bigotimes_{j=1}^{b^\mu} D^\mu_j(h)^*(W^\mu)^* FS(h), \quad h \in H
\]
from (7) and (16). Therefore,
\[
(W^\mu)^* F = \bigotimes_{j=1}^{b^\mu} D^\mu_j(h)(W^\mu)^* FS(h), \quad h \in H
\] (26)
holds by the unitarity of $D^\mu_j(h)$. Using the decomposition of $D^\mu_j(h)$ in (20), the equation (26) is rewritten as
\[
(W^\mu)^* F = \bigotimes_{j=1}^{b^\mu} \bigotimes_{\nu \in R(H)} \bigotimes_{\nu \in R(H)} D^\mu_j(h)(W^\mu)^* FS(h), \quad h \in H.
\] (27)

Therefore, with the matrix $W^\mu_M$ in (25), the equation (27) is rewritten as
\[
(W^\mu_M)^* S_M(h)(W^\mu_M)^* S_M(h), \quad h \in H.
\] (28)

Moreover, $S_M(h)$ and $S_N(h)$ are decomposed into irreducible representations by unitary matrices, say, $P_M$ of order $f$ and $P_N$ of order $r \neq f$, respectively. Note that they are defined in a similar way as for $Z$ and $W$ in (14), and $S_M$ and $S_N$ are decomposed as
\[
P_M^* S_M(h)P_M = \bigotimes_{\nu \in R(H)} \bigotimes_{\nu \in R(H)} D^\nu_{\nu\nu} \circ D^\nu_{\nu\nu}, \quad h \in H,
\] (29)
with multiplicities $b^\nu_\nu M$ and $b^\nu_\nu N$ defined in (10). Substituting (29) into (28) leads to
\[
(W^\mu_M)^* P_M = (W^\mu_M)^* S_M(h)(W^\mu_M)^* P_M
\] (30)
in this expression, the matrix $(W^\mu_M)^* P_M$ is naturally divided into some blocks as
\[
(W^\mu_M)^* P_M = \bigotimes_{\nu \in R(H)} \bigotimes_{\nu \in R(H)} D^\nu_{\nu\nu} \circ D^\nu_{\nu\nu}, \quad h \in H.
\] (31)
holds for all $\nu \in R(H)$. This is equivalent to $(W^\mu_M)^* = O$.

Combination of Lemma 1 and Lemma 2 results in Theorem 2. Moreover, if the subset $F$ defined in (13) is nonempty, the system becomes uncontrollable. Hence Theorem 1.

### 3 Examples

#### 3.1 Spherical diamond system

Consider a spherical diamond system as shown in Fig. 2(a). It is symmetric with respect to $D_6$. The dihedral group $D_m$ $(m = 1, 2, \ldots)$ of order $2m$ is defined as
\[
D_m = \{e, \rho, \ldots, \rho^{(m-1)}, \sigma, \sigma \rho, \ldots, \sigma \rho^{(m-1)}\},
\] (32)
where $\rho^m = \sigma^2 = (\sigma \rho)^2 = e$. The index set of all the irreducible representations of $D_m$, denoted as $R(D_m)$, is given by
\[
R(D_m) = \begin{cases} \{A_1, A_2, B_1, B_2, E_1, E_2, \ldots, E_{m-1}\} & \text{(m is even)}, \\ \{A_1, A_2, E_1, E_2, \ldots, E_{m-1}\} & \text{(m is odd)} \end{cases}
\] (33)
where $A_1, A_2, B_1$, and $B_2$ are one-dimensional irreducible matrix representations, and $E_i$ $(i = 1, 2, \ldots)$ are two-dimensional ones. The group $D_m$ generally represents the geometric symmetry of a regular $m$-gon.

The symmetry condition (4) holds for $T$ and $S$ defined naturally as representations of permutations. In this example, we assume that $n_0 = r_0 = 1$, hence $T = S$ follows.
The irreducible representation decomposition of $S$ of $D_6$ is described as

$$ S = 7A_1 \oplus A_2 \oplus 4B_1 \oplus 3B_2 \oplus 7E_1 \oplus 7E_2. \quad (34) $$

Two examples of the $D_3$-symmetric failures are shown in Fig. 2(b) and (c). The restrictions of the irreducible representations of $D_6$ to $D_3$ are given as

$$\begin{align*}
A_1 & \downarrow D_3 = A_1, & A_2 & \downarrow D_3 = A_2, \\
B_1 & \downarrow D_3 = A_1, & B_2 & \downarrow D_3 = A_2, \\
E_1 & \downarrow D_3 = E_1, & E_2 & \downarrow D_3 = E_1.
\end{align*} \quad (35)$$

For the failure pattern shown in Fig. 2(b) with $M = \{1, 3, 5, 7, 8, 12, 16, 32, 36, 40\}$, the representations $S_M$ and $S_N$ of $D_3$ are decomposed as

$$ S_M = 4A_1 \oplus 3E_1, \quad S_N = 7A_1 \oplus 4A_2 \oplus 11E_1. \quad (36) $$

Therefore, the condition (12) holds for $(\mu, \nu) = (A_2, A_2), (B_2, A_2)$. Consequently, Theorem 1 reveals that the system is uncontrollable.

The system shown in Fig. 2 (c) has the $D_3$-symmetric failure pattern described as $M = \{1, 20, 23, 24, 27, 28, 31, 32, 36, 40\}$. Note that the number of the functioning modules, i.e., the size of $M$ is same as that of the system in Fig. 2(b). The representations $S_M$ and $S_N$ are decomposed as

$$ S_M = 3A_1 \oplus A_2 \oplus 3E_1, \quad S_N = 8A_1 \oplus 3A_2 \oplus 11E_1. \quad (37) $$

Since the condition (12) does not hold for any pair of $\mu \in R(D_6)$ and $\nu \in R(D_3)$, the system is likely to be controllable. In fact, Theorem 3 in Section 4 reveals that the system is controllable.

### 3.2 Cubic Network

Consider a cubic system as shown in Fig. 3(a). The system is symmetric with respect to the group $O_h$ of order 48, the symmetry group of the cube. The group is obtained as the direct product of the octahedral group $O$ of order 24 and the group of reflection $Z_2 = \{e, \sigma\}$, where $\sigma^2 = e$. The 24 elements of $O$ are classified into five conjugacy classes:

- $E = \{e\}$,
- $C_2^1 = \{\text{three rotations of } \pi \text{ about fourfold axes}\}$,
- $C_2 = \{\text{six rotations of } 2\pi \text{ about twofold axes}\}$,
- $C_4 = \{\text{three rotations of } \pi/2 \text{ and three rotations of } 3\pi/2 \text{ about fourfold axes}\}$,
- $C_3 = \{\text{four rotations of } \pi/3 \text{ and four rotations of } -\pi/3 \text{ about threefold axes}\}$,

according to the notation of [6]. The index set $R(O)$ of the irreducible representations of $O$ is given by

$$ R(O) = \{A_1, A_2, E, T_1, T_2\}, $$

where $A_1$ is the one-dimensional unit representation, $A_2$ is a non-unit one-dimensional representation, $E$ is a two-dimensional irreducible representation, and $T_1$ and $T_2$ are three-dimensional ones. The products of elements in $O$ with those of $Z_2$ produces the 48 elements of $O_h$. The index set $R(O_h)$ of the irreducible representations is then given by

$$ R(O_h) = \{A_1, A_2, E, T_1, T_2, A'_1, A'_2, E', T'_1, T'_2\}. $$

The character table of the octahedral group $O_h$ is given in Table 1.

The system matrices $A$ and $B$ for the cubic system (Fig. 3 (a)) are given as

$$ A = \begin{bmatrix} PQQQOQQQ & KOOOOOGOOG \\ QPQQOQQQ & OOOOGOOOGO \\ QQPQQOOQ & KGOOGOOOGO \\ QQPQQOOQ & OOOOGOOOGO \\ QQPQQOOQ & OOOOGGOOGO \\ QQPQQOOQ & OOOOGGOOGO \end{bmatrix}, \quad B = \begin{bmatrix} KOOOOOOOOO \\ OOOOGGOOGO \\ OOOOGGOOGO \\ OOOOGGOOGO \\ OOOOGGOOGO \end{bmatrix}. $$

The symmetry condition (4) holds for $T$ and $S$ defined naturally as representations of permutations. In this example, we assume that $n_0 = r_0 = 1$, hence $T = S$ follows. The irreducible representation decomposition of $S$ of $O_h$ is described as

$$ S = A_1 \oplus T_2 \oplus A'_1 \oplus T'_2. \quad (38) $$

We deal with three cases of failures as examples: a $D_3$-symmetric failure, a $D_{2h}$-symmetric one and a $D_1$-symmetric one. The group $D_m (m = 1, 2, \ldots)$ is defined in (32) and the group $D_{2h}$ of order 8 is defined as the direct product of $D_2$ and $Z_2$. The index set $R(D_{2h})$ of the irreducible representations is given by $R(D_{2h}) =$

### Table 1: The character table of the octahedral group $O_h$.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$C_2^1$</th>
<th>$C_2$</th>
<th>$C_4$</th>
<th>$C_3$</th>
<th>$E'$</th>
<th>$C_2'$</th>
<th>$C_4'$</th>
<th>$C_3'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>$E'$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>$T_1$</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>$T_2$</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>$A'_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A'_2$</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>$E'$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>$T'_1$</td>
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<td>−1</td>
<td>1</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>$T'_2$</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 3: (a) An example of cubic systems. (b) The system retains its controllability despite the $D_3$-symmetric failures. (c) The system becomes uncontrollable because of the $D_{2h}$-symmetric failures.
\{A, B_1, B_2, B_3, A', B'_1, B'_2, B'_3\}, with the obvious meaning of primes.

An example of the first case with $D_3$-symmetric failure is shown in Fig. 3(b) with $M = \{2, 4, 5\}$ and $N = \{1, 3, 5, 7, 8\}$. The restrictions of the irreducible representations in (38) to $D_3$ are

\[
\begin{align*}
A^{\mu}_{\nu} \downarrow D_3 &= A^{\mu}_{\nu}, \\
T_2^{\mu} \downarrow D_3 &= T_2^{\mu}, \\
T_3^{\mu} \downarrow D_3 &= T_3^{\mu},
\end{align*}
\]

where $\mu^{\mu}$ denotes an index in $R(O_3)$ and $\nu^{\nu}$ denotes an index in $R(D_3)$. The superscripts to $\mu$ and $\nu$ are omitted if there is no danger of confusion. The representations $S_M$ and $S_N$ of $D_3$ are decomposed as

\[
S_M = A_1 \oplus E_1, \quad S_N = 3A_1 \oplus E_1.
\]

Since the condition (12) is not satisfied by any pair of $\mu \in R(D_3)$ and $\nu \in R(D_3)$, Theorem 1 indicates that the system is likely to be controllable. In fact, Theorem 3, which will be shown in Section 4, reveals that the system is controllable. Thus, the system turns out to retain its controllability despite the $D_3$-symmetric failure with $M = \{2, 4, 5\}$.

The second case is where the failure pattern is symmetric with respect to $D_{2h}$. An example of the $D_{2h}$-symmetric failure patterns is shown in Fig. 3(c) with $M = \{3, 4, 5, 6\}$ and $N = \{1, 2, 7, 8\}$. The restrictions of the irreducible representations in (38) to $D_{2h}$ are

\[
\begin{align*}
A^{\mu}_{\nu} \downarrow D_{2h} &= A^{\mu}_{\nu}, \\
T_2^{\mu} \downarrow D_{2h} &= T_2^{\mu}, \\
T_3^{\mu} \downarrow D_{2h} &= T_3^{\mu},
\end{align*}
\]

The representations $S_M$ and $S_N$ of $D_{2h}$ are decomposed as

\[
S_M = A_1 \oplus B_1 \oplus B'_1, \quad S_N = A_2 \oplus B_2 \oplus B'_2.
\]

Since the condition (12) holds for $(\mu, \nu) = (T_2, B_2), (T'_2, B'_2)$, Theorem 1 reveals that the system is uncontrollable.

4 Sufficient Condition for Controllability

The main theorem of the present paper (Theorem 1) shown in the previous section describes only a necessary condition for the controllability of a $G$-symmetric system after an $H$-symmetric failure. This section discusses sufficient conditions, as we have promised in Section 2. The main result of this section, giving a sufficient condition for the controllability after some failures, is shown in Theorem 3, which has been applied to determine the controllability of the systems in Figs. 2(c) and 3(b) in the previous section. We will then discuss the relationship between the sufficient condition for the controllability in Theorem 3 and the necessary condition in Theorem 1, and clarify that the condition in Theorem 1 is necessary and sufficient if $H = G$, that is, if $G$-symmetric failures occur in $G$-symmetric systems.

Our concern in this section is the controllability of the $G$-symmetric system after some failures. The symmetry of the system $(A, B)$ has been defined in terms of the equation (4) for the group $G$ and its unitary representations $T$ and $S$. We consider a generic system subject to this symmetry constraint. Since $(A, B)$ satisfies this symmetry constraint if and only if $A$ and $B$ are decomposed as (18), the genericity with respect to the symmetry condition (4) is equivalent to the genericity of the matrices $A^\mu$ and $B^\mu$ in (18) in the sense that all of their entries are independent parameters. It should be emphasized that the following discussion is based on the genericity under the symmetry.

The controllability of the $G$-symmetric system $(A, B)$ after a failure $F$ is equivalent to that of the system (23). Note that the matrices $A^\mu$ and $A^\mu$ ($\mu \neq \mu'$) have no common eigenvalues by their genericity. The discussion above, together with the well-known lemma below, leads us to Lemma 4.

**Lemma 3** Suppose the matrices $A$ and $B$ for a system $(A, B)$ are given as

\[
A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]

and square submatrices $A_1$ and $A_2$ have no common eigenvalues. Then the system $(A, B)$ is controllable if and only if two subsystems $(A_1, B_1)$ and $(A_2, B_2)$ are controllable.

**Lemma 4** A generic $G$-symmetric system $(A, B)$ remains to be controllable by a failure $F$ if and only if $(\bigoplus_{k=1}^N A^\mu, (\bigoplus_{k=1}^N B^\mu)(W^\mu)^*F)$ is controllable for each $\mu$ satisfying $a^\mu \neq 0$.

Accordingly, we are to investigate the sufficient condition for the controllability of the subsystem $(\bigoplus_{k=1}^N A^\mu, (\bigoplus_{k=1}^N B^\mu)(W^\mu)^*F)$ for $\mu$ with $a^\mu \neq 0$.

A sufficient condition for the controllability of a system $(\bigoplus_{k=1}^N A, (\bigoplus_{k=1}^N B^\mu)W^\mu)$ in general is given in the following, which serves as a key technical lemma of this section. The controllability of the system will be proved with reference to the rank of $C_m(A, B) = [A - \lambda I \mid B], \lambda \in \mathbb{C}$, the modal controllability matrix of the system $(A, B)$.

**Lemma 5** Consider a system $(A, B)$ with the matrices $A$ and $B$ given as

\[
\begin{bmatrix}
A & A \\
A & A \\
A & A \\
\end{bmatrix}, \quad B = \begin{bmatrix} B(W_{1})^* \\ B(W_{2})^* \\
\vdots \\ B(W_{N})^*\end{bmatrix},
\]

where $A \in \mathbb{C}^{a \times a}, B \in \mathbb{C}^{a \times b}$, and $W = (W^k \mid 1 \leq k \leq N) \in \mathbb{C}^{f \times bN}$ with $W^k = (w^k_j \mid 1 \leq j \leq b) \in \mathbb{C}^{f \times b}$. Suppose that $A$ and $B$ are fully dense generic matrices. Then, the system $(\bar{A}, \bar{B})$ is controllable if there exists an integer $j$ ($1 \leq j \leq b$) such that

\[
\text{rank } W_j^* = N,
\]

where $W_j^* = \begin{bmatrix} (w^1_j)^* \\ (w^2_j)^* \\
\vdots \\ (w^N_j)^*\end{bmatrix} \in \mathbb{C}^{N \times f}$.
(Proof) The modal controllability matrix of the system \((A, B)\) is given as

\[
C_m(A, B) = \begin{bmatrix}
A - \lambda I & B(W^1)^* \\
A - \lambda I & B(W^2)^* \\
\vdots & \vdots \\
A - \lambda I & B(W^N)^*
\end{bmatrix}, \quad \lambda \in \mathbb{C}.
\]

If the equation

\[
[\xi_1^*, \xi_2^*, \ldots, \xi_N^*] C_m(A, B) = O
\]

holds only for \([\xi_1^*, \xi_2^*, \ldots, \xi_N^*] = O\), where \(\xi_k \in \mathbb{C}^n (1 \leq k \leq N)\), the modal controllability matrix is of full row rank, and thus the system \((A, B)\) is controllable.

The equation (42) is equivalent to

\[
\xi_k^*(A - \lambda I) = O, \quad (1 \leq k \leq N),
\]

\[
\sum_{k=1}^N \xi_k^* B(W^k)^* = O.
\]

The equation (43) shows that \(\xi_k^* (1 \leq k \leq N)\) are left eigenvectors of \(A\) for the eigenvalue \(\lambda\). By the assumed genericity of \(A\), the eigenspace for an eigenvalue is one-dimensional. Therefore, the vectors \(\xi_k (1 \leq k \leq N)\) are given as \(\xi_k = \alpha_k \xi\) with \(\alpha_k \in \mathbb{C}\) and a nonzero vector \(\xi \in \mathbb{C}^n\) satisfying \(\xi^* A = \lambda \xi^*\). Then the equation (44) is rewritten as

\[
\sum_{k=1}^N \alpha_k \eta^*(W^k)^* = O,
\]

where \(\eta \in \mathbb{C}^b\) is given as

\[
\eta^* = \xi^* B.
\]

From the condition (41), there exists an integer \(j_0 (1 \leq j_0 < b)\) such that \(\text{rank} W_{j_0}^* = N\). Therefore, the matrix \(W_{j_0}^*\) contains \(N\) independent columns with the column indices \(\{i_1, i_2, \ldots, i_N\}\), for which

\[
det W_{j_0}^*[i_1, i_2, \ldots, i_N] \neq 0
\]

\[
\text{rank} W^*[i_1, i_2, \ldots, i_N] = N,
\]

where \(W_{j_0}^*[i_1, i_2, \ldots, i_N]\) and \(W^*[i_1, i_2, \ldots, i_N]\) denote the submatrices obtained by taking \(N\) columns indexed by \(\{i_1, i_2, \ldots, i_N\}\) of \(W_{j_0}^*\) and \(W^*\), respectively. From the equation (48), the equation (45) contains \(N\) independent equations

\[
\sum_{k=1}^N \alpha_k \eta^*(W^k)^*[i_l] = O \quad (l = 1, 2, \ldots, N).
\]

This is a system of \(N\) equations in \(N\) variables \(\{\alpha_k | 1 \leq k \leq N\}\). If \((\alpha_1, \alpha_2, \ldots, \alpha_N) \neq (0, 0, \ldots, 0)\), the coefficient matrix must be singular, i.e.,

\[
det(\eta^* (W^k)^*[i_l]) \mid 1 \leq k, l \leq N = 0
\]

holds. The left-hand side of the equation (49) is a nontrivial polynomial in \(\eta^* = (\eta_1, \eta_2, \ldots, \eta_b)\), since the coefficient of the term \(\eta_{j_0}^* \) in \(\det(\eta^*(W^k)^*[i_l])\) in (49) is equal to (47). Hence, (49) describes a nontrivial algebraic relation among the elements of \(A\) and \(B\), since \(\eta^*\) defined in (46) is obtained by the eigenvector \(\xi\) of \(A\) and the matrix \(B\). However, this is a contradiction to the genericity of \(A\) and \(B\). Therefore, \(\eta^* = 0\) which leads to \(\xi = 0\) and thus the controllability of the system \((A, B)\) is proved. ■

Thus, the following theorem, giving a sufficient condition for the controllability, is obtained.

**Theorem 3** A generic \(G\)-symmetric system \((A, B)\) retains its controllability in spite of a failure \(F\) if, for each \(\mu \in \mathbb{R}(G)\) satisfying \(a^\mu \neq 0\), there exists an integer \(j_\mu (1 \leq j_\mu \leq b^\mu)\) such that

\[
\text{rank} (W_{j_\mu}^*)^* = N^\mu.
\]

(Proof) Apply Lemma 5 to each subsystem \((\bigoplus_{k=1}^{N^\mu}, (\bigoplus_{k=1}^{N^\mu})(W_{j_\mu}^*)^*)\) corresponding to \(\mu\). ■

The condition (50) means that the projection of the basis \(W_{j_\mu}^*\) of an invariant subspace \(U_{j_\mu}^*\) in (2.3) onto the space of effective inputs \(C^l\) still serves as a basis.

In the examples in Section 3, such as Figs. 2(c) and 3(b), the systems have been shown to be controllable by Theorem 3.

It is mentioned that the converse of the statement of Lemma 5 is not always true, as follows.

**Example 1** Consider a system \((A, B)\) given as

\[
A = \alpha, \quad B = [\beta_1 \beta_2], \quad N = 3, \quad W^* = \begin{bmatrix} 100 \\ 001 \\ 010 \\ 000 \\ 100 \\ 002 \end{bmatrix}
\]

Since \(W_j^* < N\) for all \(j \in \{1, 2\}\), the condition is not satisfied. However, \(\tilde{B}\) is calculated as

\[
\tilde{B} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_1 \\ 0 & 2\beta_2 \end{bmatrix}
\]

and thus the system is generically controllable. ■

In the case of \(H = G\), the necessary condition in Theorem 1 turns out to be also sufficient for the controllability of the system.

**Theorem 4** A generic \(G\)-symmetric system \((A, B)\) remains to be controllable in spite of a \(G\)-symmetric failure \(F\) if and only if there exist no irreducible representations \(\mu\) of \(G\) such that

\[
a^\mu \neq 0, \quad b^\mu_M = 0,
\]

where \(b^\mu_M\) is the multiplicity of \(\mu\) in the representations \(S_M\) of \(G\).

(Proof) The necessity has been shown in Theorem 1.

Since the system after the failure is also \(G\)-symmetric, the representations \(S(g)\) of \(G\) are given as

\[
S(g) = S_M(g) \oplus S_N(g), \quad g \in G,
\]
where \( S_M \) and \( S_N \) are also representations of \( G \) (cf. (8)). The representations \( S_M(g) \) and \( S_N(g) \) are decomposed into a direct sum of irreducible representations as

\[
W^*_MS_M(g)W_M = \bigoplus_{\mu \in R(G)} \bigoplus_{j=1}^{b^\mu_M} D^\mu(g), \ g \in G,
\]

\[
W^*_NS_N(g)W_N = \bigoplus_{\mu \in R(G)} b^\mu_N D^\mu(g), \ g \in G,
\]

by unitary matrices \( W_M \) and \( W_N \), respectively, where the nonnegative integers \( b^\mu_M \) and \( b^\mu_N \) are the multiplicities of \( \mu \) in \( S_M \) and \( S_N \), respectively and thus \( b^\mu_M = b^\mu \) holds. Moreover, \( b^\mu_N \neq 0 \) holds for \( \mu \) satisfying \( a^\mu \neq 0 \) since there exist no irreducible representations \( \mu \) of \( G \) which satisfy (51). The controllability of the system \((A, B_F)\) after the failure \( F \) is to be proved by the controllability of the subsystems \((\bigoplus_{k=1}^{N^\mu} A^\mu, \bigoplus_{k=1}^{N^\mu} B^\mu)\) for each \( \mu \) satisfying \( a^\mu \neq 0 \) by Lemma 4.

Corresponding to the decomposition of \( S(g) \) in (52), \( W \) in (14) is given here as \( W = W_M \oplus W_N \). The matrix \( W^\mu \) is then described as

\[
W^\mu = \left[ \begin{array}{c|c} W_M^\mu & O \\ \hline O & W_N^\mu \end{array} \right],
\]

where \( W_M = (W^\mu_M \mid \mu \in R(G)) \), \( W_N = (W^\mu_N \mid \mu \in R(G)) \) with \( W_M^\mu \in \mathbb{C}^{f \times b^\mu_M N^\mu} \) and \( W_N^\mu \in \mathbb{C}^{(r-f) \times b^\mu_N N^\mu} \), and thus

\[
(W^\mu)^* F = \left[ \begin{array}{c} (W_M^\mu)^* \mu \\ O \end{array} \right].
\]

Consequently, we are to prove the controllability of the subsystem

\[
\left( \bigoplus_{k=1}^{N^\mu} A^\mu, \bigoplus_{k=1}^{N^\mu} B^\mu \right) \left( \left[ W_M^\mu \right]^* \right)
\]

on the basis of Lemma 5. Corresponding to the block structure of \( \bigoplus_{k=1}^{N^\mu} B^\mu \), the matrix \( W_M^\mu \) is divided into

\[
W_M^\mu = (w^\mu_{Mkj} \mid 1 \leq k \leq N^\mu, 1 \leq j \leq b^\mu_M),
\]

where \( w^\mu_{Mkj} \in \mathbb{C}^f \). Since the matrix \( W_M \) has been chosen to be unitary, the vectors \( \{w^\mu_{Mkj} \mid 1 \leq j \leq b^\mu_M, 1 \leq k \leq N^\mu\} \) are mutually independent. Note that \( W^\mu_{Mkj} \in \mathbb{C}^{f \times b^\mu_M} \) defined as

\[
W^\mu_{Mkj} = (w^\mu_{Mkj} \mid 1 \leq k \leq N^\mu)
\]

is the base of an invariant subspace for \( \mu \), and thus

\[
\text{rank } W^\mu_{Mkj} = N^\mu
\]

holds for all \( j (1 \leq j \leq b^\mu_M) \), where \( b^\mu_M \neq 0 \) by the assumption. Consequently, the subsystem is controllable by Lemma 5.

\section{Conclusion}

This paper has discussed the fault-tolerance of symmetric systems with respect to controllability. We consider the controllability of a system as a characteristic that should be retained despite failures in some control channels, and have revealed the underlying mathematical mechanism of the loss of controllability for symmetric systems induced by failures. The main result has been clarified in the form of a necessary condition for symmetric systems to retain their controllability in spite of symmetric failures. In the discussion, the standard results in group representation theory have been properly applied. In particular, the irreducible representations have played an important role in order to decompose the symmetric system into subsystems. Moreover, a sufficient condition for the controllability despite the symmetric failures has been also discussed based on the genericity of the subsystems. The condition has been revealed to be necessary and sufficient if \( G \)-symmetric failures occur in \( G \)-symmetric systems.

\section{Appendix: Preliminaries on Group Representation}

We describe some fundamental facts about group representations (see, for example [6]) utilized in the discussion above.

Let \( G \) be a finite group. The order of \( G \) is the number of elements in \( G \). Let \( V \) be a finite-dimensional vector space over \( \mathbb{C} \), where \( \mathbb{C} \) is the field of complex numbers, and denote by \( GL(V) \) the group of all nonsingular linear transformations of \( V \) onto itself. A representation of \( G \) on representation space \( V \) is a homomorphism \( \tau : G \rightarrow GL(V) \). The dimension of the representation is \( N = \dim V \). A subspace \( W \) of \( V \) is invariant under \( \tau \) if \( \tau(g)w \in W \) for every \( g \in G, w \in W \). A representation \( \tau \) is irreducible if the only invariant subspaces of \( V \) are \( \{0\} \) and \( V \) itself. An \( n \)-dimensional matrix representation of \( G \) is a homomorphism \( T : G \rightarrow GL(n) \), where \( GL(n) \) denotes the group of all nonsingular matrices over \( \mathbb{C} \) of order \( n \). If a basis \( \{v_1, v_2, \ldots, v_n\} \) is fixed for \( V \), we obtain a matrix representation \( T \) of \( G \) from \( \tau \). We say that \( T \) is unitary if \( T(g) \) is unitary for each \( g \in G \). Two matrix representations \( T_1 \) and \( T_2 \) of \( G \) are equivalent if there exists a nonsingular matrix \( K \) such that

\[
T_1(g) = K^{-1}T_2(g)K, \quad g \in G.
\]

We write as \( T_1 \simeq T_2 \) if \( T_1 \) and \( T_2 \) are equivalent. Any matrix representation \( T \) is equivalent to a unitary matrix representation. If \( W \) is invariant under \( \tau \), we can define a representation \( \tau^* = \tau|W \) of \( G \) on \( W \) by \( T(g)w = T(g)w, w \in W \), that we call the restriction of \( \tau \) to \( W \). The character \( \chi : G \rightarrow \mathbb{C} \) of \( \tau \) is defined by \( \chi(g) = \text{Tr} \tau(g) = \text{Tr} T(g) \).

We denote by \( \{\tau^\mu \mid \mu \in R(G)\} \) a complete list of nonequivalent irreducible representations of \( G \), where \( R(G) \) denotes an index set for the irreducible representations of \( G \). A complete list of nonequivalent irreducible matrix representations of \( G \) is denoted by \( \{D^\mu \mid \mu \in R(G)\} \). We denote the dimension of \( D^\mu \) by \( N^\mu \). Every finite-dimensional representation of a finite group can be decomposed into a direct sum.
of irreducible representations. The direct sum decomposition is obtained by
\[ V = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} V_i^\mu, \]
where \( V_i^\mu \) are invariant subspaces of \( V \) which transform irreducibly under the restrictions of \( \tau \) to \( V_i^\mu \), and \( a^\mu \) is called the multiplicity of \( \tau^\mu \) in \( \tau \). Then the matrix representation can be put into a block-diagonal form
\[ T(g) = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} D_i^\mu(g), \quad g \in G, \]
where \( D_i^\mu \) is irreducible, if we first choose a basis \( \{ v_i^\mu \mid j = 1, \ldots, N^\mu \} \) for each \( V_i^\mu \) and adopt their union as a basis of \( V \). Moreover, by choosing a basis adequately, we can have \( D_i^\mu(g) = D^\mu(g) \) (1 ≤ \( i \) ≤ \( a^\mu \)). Namely, the decomposition is as
\[ T(g) = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} D_i^\mu(g), \quad (A.1) \]
which we write as \( T = \sum_{\mu \in R(G)} a^\mu T_\mu \). The following is known as Schur’s lemma, which plays an important role to derive the main result of the present paper.

**Lemma A.1**
Let \( D_1 \) and \( D_2 \) be irreducible matrix representations of \( G \) over \( C \), and \( H \) be a matrix over \( C \). Assume that
\[ D_1(g)H = HD_2(g), \quad g \in G. \]

1. If \( D_1 \) and \( D_2 \) are not equivalent, then \( H = O \).
2. If \( D_1 = D_2 \), \( g \in G \), and \( D_1 \) is irreducible, then \( H = \alpha I \) for some \( \alpha \in C \). \( \square \)

By Schur’s lemma, the block diagonalization of the matrix \( A \) as shown in (18) is carried out as follows. The equation (4) is rewritten as
\[ (Z^*T(g)Z)(Z^*AZ) = (Z^*AZ)(Z^*T(g)Z), \quad g \in G, \]
and thus, from the equation (17),
\[ (\bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} D_i^\mu(g)) \tilde{A} = \tilde{A} (\bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} D_i^\mu(g)), \quad g \in G, \]
with \( \tilde{A} = Z^*AZ = (\{ Z_i^\mu \}^* AZ_i^\mu \mid \mu, \mu' \in R(G)) \). From the first statement of Schur’s lemma,
\[ (Z_i^\mu)^* AZ_i^\mu = O, \quad \text{if} \ \mu \neq \mu'. \]
Therefore,
\[ \tilde{A} = \bigoplus_{\mu \in R(G)} \tilde{A}_\mu, \quad (A.2) \]
where \( \tilde{A}_\mu = (Z_i^\mu)^* AZ_i^\mu \in \mathbb{C}(a^\mu N^\mu \times a^\mu N^\mu) \). Moreover, from the second statement of Schur’s lemma,
\[ (Z_i^\mu)^* AZ_j^\mu = \alpha_{ij} I_{N^\mu}, \]
with \( 1 \leq i, j \leq a^\mu \) for some \( \alpha_{ij} \in \mathbb{C} \). Therefore, the matrix \( \tilde{A}_\mu \) is equal, up to permutations of rows and columns, to a block-diagonal matrix consisting of \( N^\mu \) copies of an identical matrix \( A^\mu = (\alpha_{ij}^\mu) \) of size \( a^\mu \times a^\mu \). To sum up,
\[ \tilde{A} = \bigoplus_{\mu \in R(G)} \bigoplus_{k=1}^{N^\mu} A^\mu, \quad (A.3) \]
is derived as in (18). Similar calculation leads to the block diagonalization of \( B \) as in (18).

A subgroup \( H \) of \( G \) is a subset of \( G \) which is itself a group under the operation in \( G \). If \( T \) is a representation of \( G \) on \( V \), we can obtain a representation \( T_H \) of a subgroup \( H \) of \( G \) by restricting \( T \) to \( H \),
\[ T_H(h) = T(h), \quad h \in H, \]
and we write \( T_H = T \downarrow H \). Similarly, \( \mu \downarrow H \) means the restriction of the irreducible representation \( \mu \) of \( G \).

**References**