How to reconstruct a gradient for parabolic systems

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Keywords: Parabolic System, Regional Gradient Observability, Gradient Strategic Sensor, Gradient Reconstruction.

Abstract

The aim of this paper is to develop the concept of regional gradient observability for parabolic systems. This new concept is closer to practical considerations since it only requires the observation of the gradient on a subregion of the system domain. First we give definitions and characterizations of this concept and we establish necessary conditions for sensors structure in order to obtain regional gradient observability. Then we give an approach which achieves regional gradient reconstruction. The developed method is original and leads to numerical algorithm illustrated by simulations.

1 Introduction

The analysis of distributed systems consists of a set of concepts as controllability, observability, stability, stabilisability, detectability,...see [3,8]that allows a better understanding of those systems and consequently enables to conduct them in better way. This analysis may be realized by means of the output and input parameters structure see [7]. Recently the concept of regional analysis has been introduced by [4,6] which give important tools for solving many real problems, particularly the regional observability concept which refers to problems in which the observed state of interest is not fully specified as a state, but refers only to a region \( \omega \), a portion of the spatial domain \( \Omega \) on which the system is considered, and has been extended by [1] to the case where the subregion \( \omega \) is a part of the boundary \( \partial \Omega \) of \( \Omega \).

An extention of the observability concept and that is very important in practical application is that of regional gradient observability. For the dual of this concept see [2].

In this paper we introduce the regional gradient observability concept for parabolic systems. That is to say one may be concerned with the knowledge of the state gradient only in a critical subregion. The introduction of this concept is motivated by real situations. This is the case, for example, of energy exchange problem, where the aim is to determine the energy exchanged between a casting plasma on a plane target which is perpendicular to the direction of the flow from measurements carried out by internal thermocouples (fig. 1). More precisely let \((S)\) be a linear system with suitable state space, and suppose that the initial state \( y_0 \) and its gradient \( \nabla y_0 \) are unknowns. Suppose now that measurements are given by means of an output function (depending on the number and the structure of the sensors). The problem concerns the reconstruction of the initial gradient \( \nabla y_0 \) on the subregion \( \omega \) of the system domain \( \Omega \).

The paper is organized as follows: The second section is devoted to the presentation of the considered system, definitions and characterizations of this new concept. In the third section we establish a relation between the regional gradient observability and sensors. The fourth section is focused in the regional reconstruction of the initial state gradient, various types of sensors are discussed. In the last section we develop a numerical approach which is illustrated by simulations that lead to some conjectures.

2 Regional gradient observability

2-1 Problem statement

Let \( \Omega \) be an open bounded regular subset of \( \mathbb{R}^n \) \((n = 1, 2, 3)\) with a smooth boundary \( \partial \Omega \) and \( \omega \) a subregion of \( \Omega \). For \( T > 0 \), we denote \( Q = \Omega \times [0, T], \Sigma = \partial \Omega \times [0, T] \).

We consider a parabolic system defined by

\[
\begin{cases}
\frac{\partial y}{\partial t}(x, t) = Ay(x, t) & Q \\
y(\xi, t) = 0 & \Sigma \\
y(x, 0) = y_0(x) & \Omega 
\end{cases}
\]

where \( A \) is a second order differential linear operator with a compact resolvent and generates a strongly continuous
A sensor is defined by a couple \((D, f)\), where \(D\) is the spatial support of the sensor represented by a non empty part of \(\Omega\) and \(f\) represents the distribution of the sensing measurements on \(D\).

Depending on the nature of \(D\) and \(f\), one could have different types of sensors:

- it may be pointwise if \(D = \{b\}\) with \(b \in \overline{\Omega}\) and \(f = \delta_i(-b)\), where \(\delta\) is the Dirac mass concentrated in \(b\).

The output function (2) may be written in the form

\[ z(t) = y(b, t) \tag{3} \]

it may be zone when \(D \subset \overline{\Omega}\) and \(f \in L^2(D)\). The output function (2) may be written in the form

\[ z(t) = \int_D y(x, t) f(x) dx \tag{4} \]

For the definitions and properties of strategic and regional strategic sensors, we refer to [6,7].

The observation space is \(O = L^2(0, T; \mathbb{R}^q)\). The system (1) is autonomous and (2) allows to write \(z(t) = CS(t)y_0(x)\). We define the operator

\[ K : \quad X \rightarrow O \quad h \rightarrow CS(\cdot \cdot \cdot)h \]

which is, in the zone case, linear and bounded with the adjoint denoted by \(K^*\).

The operator

\[ \nabla : \quad H^1(\Omega) \rightarrow (L^2(\Omega))^n \]

\[ y \rightarrow \nabla y = (\frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}) \]

while \(\nabla^*\) denotes its adjoint.

For a non empty subset \(\omega\) of \(\Omega\), we consider also the operators:

\[ \chi_{\omega} : \quad (L^2(\Omega))^n \rightarrow (L^2(\omega))^n \]

\[ y \rightarrow y_{|\omega} \]

and

\[ \tilde{\chi}_{\omega} : \quad L^2(\Omega) \rightarrow L^2(\omega) \]

\[ y \rightarrow y_{|\omega} \]

Their adjoints are denoted by \(\chi_{\omega}^*\) and \(\tilde{\chi}_{\omega}^*\).

We finally introduce the operator \(H = \chi_{\omega} \nabla K^*\) from \(O\) into \((L^2(\omega))^n\).

**Definition 2.1**

The system (1) together with the output (2) is said to be exactly (resp. approximately) regionally gradient observable on \(\omega\) if \(ImH = (L^2(\omega))^n\) (resp. \(ImH = (L^2(\omega))^n\)), a such system will be said \(G\)-observable on \(\omega\).

It is clear that:

- If a system is exactly (resp. approximately) regionally observable on \(\omega\) see [5] then it is exactly (resp. approximately) regionally \(G\)-observable on \(\omega\).

- If a system is exactly (resp. approximately) \(G\)-observable on \(\omega\), it is exactly (resp. approximately) \(G\)-observable on \(\omega_1 \subset \omega\), but the following example gives gradient which may be regionally \(G\)-observable on \(\omega\) but not \(G\)-observable on the whole domain \(\Omega\).

**2-2 Counter example**

Let \(\Omega = [0, 1] \times [0, 1]\) and consider the following bidimensional system:

\[
\begin{aligned}
\frac{\partial y}{\partial t}(x_1, x_2, t) &= \frac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) \\
y(\xi, \eta, t) &= 0 \\
y(x_1, x_2, 0) &= y_0(x_1, x_2)
\end{aligned}
\tag{5}
\]

The measurements are given by the output

\[ z(t) = \int_D y(x_1, x_2, t) f(x_1, x_2) dx_1 dx_2 \]

where \(D = \{\frac{1}{2}\} \times [0, 1]\) is the sensor support and \(f(x_1, x_2) = \sin \pi x_2\) is the function measure.

Let \(\omega = [0, 1] \times \frac{1}{6} \leq \frac{1}{2} \) and

\[ g(x_1, x_2) = (\cos \pi x_1 \sin 2\pi x_2, \sin \pi x_1 \cos 2\pi x_2) \in (L^2(\Omega))^2 \]

be the gradient to be observed.

The operator \(A = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\) generates a semi-group \((S(t))_{t \geq 0}\) in \(H^1(\Omega)\) given by:

\[ S(t)y = \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t} y_i < \varphi_{ij} >_{H^1(\omega)} \varphi_{ij} \]

where \(\varphi_{ij}(x_1, x_2) = 2a_{ij} \sin(i\pi x_1) \sin(j\pi x_2)\)

\[ \lambda_{ij} = -(i^2 + j^2)\pi^2 \quad \text{and} \quad a_{ij} = (1 - \lambda_{ij})^{-\frac{1}{2}} \]

**Proposition 2.2**

The gradient \(g\) is not approximately \(G\)-observable on the whole domain \(\Omega\), however it is approximately \(G\)-observable on the subregion \(\omega\).

**Proof**

To prove that \(g\) is not \(G\)-observable on \(\Omega\), we must show that \(g \in Ker \nabla^*\).
\[ K \nabla^* (g) \]
\[
= \sum_{i,j=1}^{\infty} e^{\lambda_{ij} t} < \nabla^* g, \varphi_{ij} >_{H^2(\Omega)} < \varphi_{ij}, f >_{L^2(D)} \\
= 4 \sum_{i,j=1}^{\infty} e^{\lambda_{ij} t} \int_0^1 \cos i\pi x_1 \cos \pi x_1 dx_1 \times \\
\int_0^1 \sin j\pi x_2 \sin 2\pi x_2 dx_2 \\
+ 2 j\pi a_{ij} \int_0^1 \sin i\pi x_1 \sin \pi x_1 dx_1 \times \\
\int_0^1 \cos j\pi x_2 \cos 2\pi x_2 dx_2 \\
+ 2 a_{ij} \int_0^{\frac{i}{2}} \sin i\pi x_2 \sin 2\pi x_2 dx_2 \\
\frac{(\sqrt{3} - 1)}{2(1 + 2\pi^2)} e^{-2\pi^2 t} \neq 0 \blacksquare \\
\]

Now, we show that the restriction of \( g \) to the subregion \( \omega \) is \( G \)-observable on \( \omega \).

\[ K \nabla^* \chi_\omega^* \chi_\omega (g) \]
\[
= \sum_{i,j=1}^{\infty} e^{\lambda_{ij} t} < \nabla^* \chi_\omega^* \chi_\omega g, \varphi_{ij} >_{H^2(\Omega)} < \varphi_{ij}, f >_{L^2(D)} \\
= 4 \sum_{i,j=1}^{\infty} e^{\lambda_{ij} t} \int_0^1 \cos i\pi x_1 \cos \pi x_1 dx_1 \times \\
\int_0^1 \sin j\pi x_2 \sin 2\pi x_2 dx_2 \\
+ 2 j\pi a_{ij} \int_0^1 \sin i\pi x_1 \sin \pi x_1 dx_1 \times \\
\int_0^1 \cos j\pi x_2 \cos 2\pi x_2 dx_2 \\
+ 2 a_{ij} \int_0^{\frac{i}{2}} \sin i\pi x_2 \sin 2\pi x_2 dx_2 \\
\frac{(\sqrt{3} - 1)}{2(1 + 2\pi^2)} e^{-2\pi^2 t} \neq 0 \blacksquare \\
\]

**2-3 Characterizations**

We may characterize the \( G \)-observability on \( \omega \) by the following results.

**Proposition 2.3**

1. The system (1) together with the output (2) is exactly \( G \)-observable on \( \omega \) if and only if for all \( z^* \in (L^2(\omega))^n \), there exists \( c > 0 \) such that:

\[ ||z^*||_{(L^2(\omega))^n} \leq c ||K \nabla^* \chi_\omega^* z^*||_\sigma \]

2. The system (1) together with the output (2) is approximately \( G \)-observable on \( \omega \) if and only if the operator \( N_\omega = HH^* \) is positive definite.

**Proof**

1) Let \( h = Id_{(L^2(\omega))^n} \) and \( g = \chi_\omega \nabla K^* \). Since the system is exactly \( G \)-observable on \( \omega \), we have \( \text{Im} h \subset \text{Im} g \) which is equivalent to \( \text{There exists } c > 0 \) such that

\[ ||h z^*||_{(L^2(\omega))^n} \leq c ||g z^*||_{L^2(0, T ; \mathbb{R}^n)} \forall z^* \in (L^2(\omega))^n \]

2) \( \Rightarrow \) Let \( z^* \in (L^2(\omega))^n \) such that

\[ < N_\omega z^*, z^* >_{(L^2(\omega))^n} = 0 \]

therefore \( z^* \in \ker H^* \) and consequently \( z^* = 0 \). That is \( N_\omega \) is positive definite.

Conversely let \( z^* \in (L^2(\omega))^n \) such that \( H^* z^* = 0 \), then \( H^* z^*, H^* z^* >_\sigma = 0 \) and thus

\[ < N_\omega z^*, z^* >_{(L^2(\omega))^n} = 0 \]

therefore \( z^* = 0 \). That is the system is approximately \( G \)-observable. \( \blacksquare \)

Now, assume that the system (1) is not \( G \)-observable on \( \Omega \) and let \((\varphi_i)_{i \in \mathbb{N}^n} \) be a basis in \((L^2(\Omega))^n\).

Let \( I \subset \mathbb{N}^n \) such that \( \ker K \nabla^* = \text{span}\{(\varphi_i)_{i \in I}\} \) and \( J = \mathbb{N}^n \setminus I \). We have

**Proposition 2.4**

The following properties are equivalent

1. The system (1) is approximately \( G \)-observable on \( \omega \).
2. \( \text{span}\{(\chi_\omega \varphi_i)_{i \in J}\} = (L^2(\omega))^n \).
3. If \( y \in (L^2(\omega))^n \) such that \( < y, \chi_\omega \varphi_i >_{(L^2(\omega))^n} = 0 \) for all \( i \in J \) then \( y = 0 \).
4. If \( \sum a_i \varphi_i = 0 \) on \( \Omega \) then \( a_i = 0 \) for all \( i \in I \).

**Proof**

(1) \( \Rightarrow \) Let \( y \in (L^2(\omega))^n \) then for \( \varepsilon > 0 \) there exists \( z \in \mathcal{O} \) such that

\[ ||y - \chi_\omega \nabla K^* z||_{(L^2(\omega))^n} \leq \varepsilon \]

but

\[ \nabla K^* z = \sum_{i \in \mathbb{N}^n} < \nabla K^* \varphi_i >_{(L^2(\Omega))^n} \varphi_i \]

Thus \( \chi_\omega \nabla K^* z = \sum_{i \in J} < z, K \nabla^* \varphi_i >_\sigma \chi_\omega \varphi_i \)

then

\[ ||y - \sum_{i \in J} < z, K \nabla^* \varphi_i >_\sigma \chi_\omega \varphi_i||_{(L^2(\omega))^n} \leq \varepsilon \]

hence \( y \in \{\chi_\omega \varphi_i\}_{i \in J} \)

(2) \( \Rightarrow \) Let \( y \in (L^2(\omega))^n \) from 2), we have that for all \( \varepsilon > 0 \), there exists \( a_j \) \( (j \in J) \) such that

\[ ||y - \sum_{j \in J} a_j \chi_\omega \varphi_j||_{(L^2(\omega))^n} < \varepsilon \]

with \( < y, \chi_\omega \varphi_j >_{(L^2(\omega))^n} = 0 \) \( \forall j \in J \)

we deduce and

\[ ||y||_{(L^2(\omega))^n} < \varepsilon \]

for all \( \varepsilon > 0 \) thus \( y = 0 \).

(3) \( \Rightarrow \) Let

\[ \sum_{i \in I} a_i \varphi_i = 0 \] on \( \Omega \) and consider

\[ y = \chi_\omega \left( \sum_{i \in I} a_i \varphi_i \right). \] For \( j \in J \), we have

\[ < y, \chi_\omega \varphi_j >_{(L^2(\omega))^n} = \sum_{i \in I} a_i < \varphi_i, \varphi_j >_{(L^2(\omega))^n} = 0 \].

(From 3) \( y = 0 \), hence

\[ \sum_{i \in I} a_i \varphi_i = 0 \text{ on } \Omega \]
and $a_i = 0 \forall i \in I$.

(4)$\Rightarrow$(1)

Let $y \in (L^2(\omega))^n$ such that $K\nabla^* \chi_n^* y = 0$, we have $\chi_n^* y \in (L^2(\Omega))^n$ then

$$K\nabla^* \chi_n^* y = K\nabla^* (\sum_{i \in I} y_i \chi_i \overline{\varphi}_i > (L^2(\omega))^n \overline{\varphi}_i)$$

$$= K\nabla^* (\sum_{i \in J} y_i \chi_i \overline{\varphi}_i > (L^2(\omega))^n \overline{\varphi}_i) = 0$$

therefore

$$\sum_{j \in J} y_j \chi_j \overline{\varphi}_j > (L^2(\omega))^n \overline{\varphi}_j \in \text{span}\{(\varphi_i)_{i \in I}\}$$

thus

$$\sum_{j \in J} y_j \chi_j \overline{\varphi}_j > (L^2(\omega))^n = 0 \forall j \in J$$

then

$$\chi_n^* y = \sum_{i \in I} y_i \chi_i \overline{\varphi}_i > (L^2(\omega))^n \overline{\varphi}_i = 0 \text{ on } \Omega \setminus \omega$$

and from 4), $\sum_{i \in I} y_i \chi_i \overline{\varphi}_i > (L^2(\omega))^n = 0 \forall i \in I$

hence $y = 0$. □

By the property (4) of the proposition(2-4) it is easy to show the result

**Corollary 2.5**

**Under the hypothesis of the proposition (2-4), the system (1) is approximately $G$-observable on all $\omega \subset \Omega$ such that $\sum_{i \in I} \chi_i \overline{\varphi}_i > (L^2(\omega))^n = 0 \forall i, j \in I, i \neq j$.**

### 3 Gradient strategic sensors

The purpose of this section is to link the regional gradient observability with the sensors structure. Let us consider the system (1) observed by $q$ sensors $(D_i, f_i)$ which may be pointwise or zone.

**Definition 3.1**

A sensor $(D, f)$ (or a sequence of sensors) is said to be gradient strategic on $\omega$ if the observed system is $G$-observable on $\omega$, a such sensor will be said $G$-strategic on $\omega$.

We assume that the operator $A$ has a complete set of eigenvalues in $H^1(\Omega)$ denoted by $(\varphi_i)$ orthonormal in $L^2(\omega)$ and the associated eigenvalues $\lambda_i$ are of multiplicity $r_i$. Assume also that $r = \sup_{i \in I} r_i$ is finite and $A$ is of constant coefficients, then we have the following result :

**Proposition 3.2**

If the sequence of sensors $(D_k, f_k)_{1 \leq k \leq q}$ is $G$-strategic on $\omega$, then $q \geq r$ and $\text{rank} M_m = r_m$ where,

$$(M_m)_{i,j} = \begin{cases} n \sum_{k=1}^n \frac{\partial \varphi_{m_j}}{\partial x_k} (b_i) & \text{pointwise case} \\ \sum_{k=1}^n \varphi_{m_j} \partial \varphi_{m_j} < f_i > (L^2(D_i)) & \text{zone case} \end{cases}$$

$1 \leq i \leq q$ and $1 \leq j \leq r_m$

**Proof**

The proof is developed in the case of zone sensors.

The sequence of sensors $(D_k, f_k)_{1 \leq k \leq q}$ is $G$-strategic on $\omega$ if and only if

$$\{z \in (L^2(\omega))^n | < Hu, z > (L^2(\omega))^n = 0 \forall u \in G \} \Rightarrow z = 0$$

Suppose that the sequence of sensors $(D_k, f_k)_{1 \leq k \leq q}$ is $G$-strategic on $\omega$ but for a certain $m \in \mathbb{N}$ $\text{rank} M_m \neq r_m$, then there exists

$$z_m = (z_{m_1}, ..., z_{m_r})^T \neq 0 \text{ such that } M_m z_m = 0.$$

Let $z_0 = \sum_{j=1}^{r_m} z_{m_j} \varphi_{m_j}$, $Z_0 = (z_0, ..., z_0)$

and $\varphi_{z_p} = (z_0, \varphi_{p_1}, ..., z_0, \varphi_{p_q})^T$

then

$$< Hu, Z_0 > (L^2(\omega))^n = \sum_{k=1}^n y_k \frac{\partial}{\partial x_k} (K^* u), z_0 > L^2(\omega)$$

$$= \sum_{k=1}^n \frac{\partial}{\partial x_k} (\tilde{\varphi}(T)), \tilde{\chi}_n^* z_0 > L^2(\Omega)$$

where $\tilde{\varphi}$ is the solution of the following system :

$$\begin{cases} \frac{\partial \tilde{\varphi}}{\partial t}(x, t) = A^* \tilde{\varphi}(x, t) + \sum_{i=1}^q f_i \chi_{D_i} u_i(T - t) & Q \\ \tilde{\varphi}(\xi, t) = 0 & \Sigma \\ \tilde{\varphi}(x, 0) = 0 & \Omega \end{cases}$$

assumed to be in $L^2(0, T; H^2_0(\Omega))$.

Consider the system

$$\begin{cases} \frac{\partial \varphi}{\partial t}(x, t) = -A \varphi(x, t) & Q \\ \varphi(\xi, t) = 0 & \Sigma \\ \varphi(x, T) = \tilde{\chi}_n^* z_0 & \Omega \end{cases}$$

and assume that the solution $\varphi$ is in $L^2(0, T; H^2_0(\Omega))$.

$\cap C^1(\Xi \times ]0, T[)$.

Multiply (6) by $\frac{\partial \varphi}{\partial x_k}$ and integrate on $Q$, we obtain

$$\int_Q \frac{\partial \varphi}{\partial x_k}(x, t) \frac{\partial \tilde{\varphi}}{\partial t}(x, t) dx dt$$

$$= \int_Q A^* \tilde{\varphi}(x, t) \frac{\partial \varphi}{\partial x_k}(x, t) dx dt$$

$$+ \int_Q \sum_{i=1}^q f_i \chi_{D_i} u_i(T - t) \frac{\partial \varphi}{\partial x_k}(x, t) dx dt$$
but we have
\[
\int_Q \frac{\partial \varphi}{\partial x_k} (x, t) \frac{\partial \tilde{y}}{\partial t} (x, t) dx dt = \int_\Omega \left[ \frac{\partial \varphi}{\partial x_k} (x, t) \tilde{y}(x, t) \right]_0^T dx \\
+ \int_Q A \left( \frac{\partial \varphi}{\partial x_k} (x, t) \tilde{y}(x, t) \right) dx dt \\
= \int_Q \frac{\partial \varphi}{\partial x_k} (x, T) \tilde{y}(x, T) dx \\
+ \int_Q A \left( \frac{\partial \varphi}{\partial x_k} (x, t) \right) \tilde{y}(x, t) dx dt.
\]

then
\[
\int_\Omega \frac{\partial \varphi}{\partial x_k} (x, T) \tilde{y}(x, T) dx \\
= - \int_Q A \left( \frac{\partial \varphi}{\partial x_k} (x, t) \right) \tilde{y}(x, t) dx dt \\
\times \int_Q A^* \left( \frac{\partial \varphi}{\partial x_k} (x, t) \right) \tilde{y}(x, t) dx dt \\
+ \int_Q \left( \sum_{i=1}^q f_i \chi_{D_i} u_i (T - t) \right) \frac{\partial \varphi}{\partial x_k} (x, t) dx dt \\
= - \int Q \left( \sum_{i=1}^q f_i \chi_{D_i} u_i (T - t) \right) \frac{\partial \varphi}{\partial x_k} (x, t) dx dt \\
+ \sum_{\Omega} \int_{\Omega} \tilde{q} (\zeta, t) \frac{\partial \varphi}{\partial x_k} (\zeta, t) d\zeta dt \\
+ \int_{\Omega} \sum_{i=1}^q f_i \chi_{D_i} u_i (T - t) \frac{\partial \varphi}{\partial x_k} (x, t) dx dt
\]

and since \( \tilde{y} (\zeta, t) \in L^2 (0; T; H_0^1 (\Omega)) \) we have :
\[
\int_\Omega \frac{\partial \varphi}{\partial x_k} (x, T) \tilde{y}(x, T) dx \\
= \int Q \left( \sum_{i=1}^q f_i \chi_{D_i} u_i (T - t) \right) \frac{\partial \varphi}{\partial x_k} (x, t) dx dt
\]

thus
\[
\int_\Omega \varphi (x, T) \frac{\partial \tilde{y}}{\partial x_k} (x, t) dx \\
= - \sum_{i=1}^q \int_0^T < f_i, \frac{\partial \varphi}{\partial x_k} >_{L^2 (D_i)} u_i (T - t) dt \\
\]

and we have
\[
< \chi_\omega \nabla K^* u, Z_0 >_{(L^2 (\Omega))^n} \\
= \sum_{k=1}^n \int_\Omega \frac{\partial \tilde{y}}{\partial x_k} (x, T) \varphi (x, T) dx \\
= - \sum_{i=1}^q \int_0^T \sum_{j=1}^p < f_i, \frac{\partial \varphi}{\partial x_k} >_{L^2 (D_i)} u_i (T - t) dt \\
\]

but
\[
\varphi (x, t) = \sum_{p=1}^\infty e^{-\lambda_p (t - T)} \sum_{j=1}^r < z_0, \varphi_{p_j} >_{L^2 (\omega)} \varphi_{p_j}
\]

then
\[
\sum_{k=1}^n < f_i, \frac{\partial \varphi}{\partial x_k} >_{L^2 (D_i)} \\
= \sum_{p=1}^\infty e^{-\lambda_p (T - t)} \sum_{j=1}^r < z_0, \varphi_{p_j} >_{L^2 (\omega)} \\
\times \sum_{k=1}^\infty < \varphi_{p_k}, f_i >_{L^2 (D_i)} \\
= \sum_{p=1}^\infty e^{-\lambda (T - t)} (M_p z_p),
\]

therefore
\[
\left\{ \begin{array}{l}
< \chi^\omega \nabla K^* u, Z_0 >_{(L^2 (\omega))^n} \\
= - \sum_{i=1}^q \sum_{p=1}^\infty e^{-\lambda_p (T - t)} (M_p z_p), u_i (T - t) dt 
\end{array} \right.
\]

thus
\[
< \chi^\omega \nabla K^* u, Z_0 >_{(L^2 (\omega))^n} \\
= - \sum_{i=1}^q \int_0^T e^{-\lambda m (T - t)} (M_m z_m), u_i (T - t) dt = 0.
\]

This is true for all \( u \in L^2 (0; T; \mathbb{R}^q) \), then
\( Z_0 \in \text{Ker} (H^*) \) which contradicts the assumption that the sequence of sensors is \( G \)-strategic. \( \blacksquare \)

It's from formula (8) we deduce the result bellow.

**Corollary 3.3**

In the monodimensional case, the system (1) is \( G \)-observable if and only if \( q \geq r \) and \( \text{rank} M_m = r_m \) where \( M_m \) is given in the previous proposition.

**Application**

Consider the system described by the parabolic equation
\[
\begin{cases}
\frac{\partial y}{\partial t} (x, t) - \frac{\partial^2 y}{\partial x^2} (x, t) = 0 & \text{on } [0, 1] \times [0, T] \\
y (0, t) = y (1, t) = 0 & \text{on } [0, T] \\
y (x, 0) = y_0 (x) & \text{on } [0, 1]
\end{cases}
\]

with the output \( z (t) = y (b, t), b \in [0, 1] \) and \( t \in [0, T] \).

The previous system is observable on \( [0, 1] \), see [7], if and only if \( b \notin S = \bigcup_{m=1}^\infty \{ \frac{k}{m} \mid k \in [1, m - 1] \cap \mathbb{N} \} \).

It is \( G \)-observable on \( [0, 1] \) if and only if \( b \notin S_G = \bigcup_{m=1}^\infty \{ \frac{2k + 1}{2m} \mid k \in [0, m - 1] \cap \mathbb{N} \} \).

We have \( S_G \subset S \), which shows that there exist Sensors which are \( G \)-strategic but not strategic.

4 Regional gradient reconstruction

In this section we give an approach which allows the reconstruction of the initial state gradient on \( \omega \) of the system (1).
This approach does not take into account what must be the residual initial gradient on the subregion \( \omega^c = \Omega \setminus \omega \). Consider the sets:

\[
\mathcal{F} = \{ h \in (L^2(\Omega))^n \mid h = 0 \text{ on } \omega \} \cap \{ \nabla f \mid f \in H^1_0(\Omega) \}
\]

\[
\tilde{\mathcal{F}} = \{ h \in (L^2(\Omega))^n \mid h = 0 \text{ on } \omega^c \} \cap \{ \nabla f \mid f \in H^1_0(\Omega) \}.
\]

For \( \phi_0 \in H^1_0(\Omega) \), consider the system

\[
\begin{align*}
\frac{\partial \phi}{\partial t}(x, t) &= A\phi(x, t) \quad Q \\
\phi(x, 0) &= 0 \quad \Sigma \\
\phi(x, t) &= \phi_0(x) \quad \Omega
\end{align*}
\]  

and assume that (10) has a unique solution

\[
\phi \in L^2(0, T; H^1(\Omega)) \cap C^1(\Omega \times [0, T]) \cap C(\Omega \times [0, T]).
\]

### 4.1 Pointwise sensor case

We consider the system (1) with the output function (3). For \( \tilde{\phi}_0 \in \tilde{\mathcal{F}} \), there exists a unique \( \phi_0 \in H^1_0(\Omega) \) such that \( \tilde{\phi}_0 = \nabla \phi_0 \), then we consider a semi-norm on \( \tilde{\mathcal{F}} \) defined by:

\[
\tilde{\phi}_0 \rightarrow \|\tilde{\phi}_0\|_{\tilde{\mathcal{F}}}^2 = \int_0^T \left( \sum_{k=1}^n \frac{\partial \phi}{\partial x_k} b(t) \right)^2 dt
\]  

(11)

where \( \phi \) is the solution of (10) and assume that the solution \( \psi_1 \) of the equation

\[
\begin{align*}
\frac{\partial \psi_1}{\partial t}(x, t) &= -A^* \psi_1(x, t) - \sum_{k=1}^n \frac{\partial \phi}{\partial x_k} (b(t)) \delta(x - b) \\
\psi_1(x, 0) &= 0 \quad \Omega
\end{align*}
\]  

is in \( L^2(0, T; H^1_0(\Omega)) \).

When the semi-norm (11) is a norm, we denote also by \( \tilde{\mathcal{F}} \) the completion of \( \mathcal{F} \) and we consider the operator

\[
\Lambda : \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}^*
\]

\[
\phi_0 \rightarrow P(\Psi(0))
\]

where \( P = \chi_+ \chi_- \) and \( \Psi(0) = (\psi_1(0), ..., \psi_1(0)) \).

Introduce the system:

\[
\begin{align*}
\frac{\partial \phi}{\partial t}(x, t) &= -A^* \psi_1(x, t) - \sum_{k=1}^n \frac{\partial \phi}{\partial x_k} (b(t)) \delta(x - b) \\
\frac{\partial \psi}{\partial x_A^*}(x, t) &= 0 \\
\psi(x, T) &= 0
\end{align*}
\]  

(13)

If \( \tilde{\phi}_0 \) is chosen such that \( \psi_1(0) = \psi(0) \) on \( \omega \) then the system (13) looks like the adjoint of the system (1), and then the regional gradient observability turns up to solve the equation

\[
\Lambda(\tilde{\phi}_0) = P(\Psi(0))
\]

(14)

where \( \Psi(0) = (\psi(0), ..., \psi(0)) \) with \( \psi \) is the solution of (13).

### Proposition 4.1

If the sensor \((b, b_0)\) is \( G \)-strategic on \( \omega \) then the equation (14) has a unique solution \( \tilde{\phi}_0 \) which is the initial state gradient to be observed on \( \omega \).

**Proof**

(a) Let us show first that if the system (1) is \( G \)-observable then (11) defines a norm on \( \tilde{\mathcal{F}} \).

Consider a basis \((\varphi_i)\) of the eigenfunctions of \( A \), without loss of generality we suppose that the associated eigenvalues \( \lambda_i \) are of multiplicity one.

The mapping (11) defines a norm on \( \tilde{\mathcal{F}} \), indeed

\[
\|\tilde{\phi}_0\|_{\tilde{\mathcal{F}}} = 0 \quad \Rightarrow \quad \sum_{i=1}^{\infty} e^{\lambda_i t} \varphi_i > H^1(\Omega) \sum_{k=1}^n \frac{\partial \varphi_i}{\partial x_k}(b) = 0 \quad \text{a.e. on} \quad (0, T)
\]

Then \( \varphi_i > H^1(\Omega) \sum_{k=1}^n \frac{\partial \varphi_i}{\partial x_k}(b) = 0 \quad \forall i \) and since the sensor \((b, b_0)\) is \( G \)-strategic on \( \omega \) (proposition (3-2)), we obtain

\[
\sum_{k=1}^n \frac{\partial \varphi_i}{\partial x_k}(b) \neq 0 \quad \forall i
\]

Consequently \( \phi_0 = 0 \) and thus \( \tilde{\phi}_0 = 0 \).

Conversely, \( \tilde{\phi}_0 = 0 \) implies \( \phi_0 = c \) (constant), since \( \phi \in C(\Omega \times [0, T]) \) and from \( \phi = 0 \) in \( \Sigma \), the result follows.

(b) Let us prove now that (14) has a unique solution.

The equation (14) has a unique solution because the operator \( \Lambda \) is an isomorphism. Indeed multiply the equation (12) by \( \frac{\partial \phi}{\partial x_k} \) and integrate on \( Q \), we obtain

\[
\Lambda(\tilde{\phi}_0) = P(\Psi(0))
\]

(14)

which gives

\[
\begin{align*}
&- A^* \psi_1(x, t) \frac{\partial \phi}{\partial x_k}(b(t)) \delta(x - b) > L^2(\Omega)
\end{align*}
\]  

(14)
and with the finale condition we obtain
\[
- < \frac{\partial \phi}{\partial x_k}(x,0), \psi_1(x,0) >_{L^2(\Omega)} \\
= \frac{1}{\partial x_k} A \frac{\partial \phi}{\partial x_k}(x,t), \psi_1(x,t) >_{L^2(Q)} \\
- < \frac{1}{\partial x_k} \phi(x,t), A^* \psi_1(x,t) >_{L^2(Q)} \\
- \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(b,t)dt 
\]
Using Green formula, we obtain
\[
< \frac{\partial \phi}{\partial x_k}(x,0), \psi_1(x,0) >_{L^2(\Omega)} \\
= \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(b,t)dt 
\]
then
\[
\sum_{k=1}^n < \frac{\partial \phi_0}{\partial x_k}(x), \psi_1(x,0) >_{L^2(\Omega)} \\
= \sum_{k=1}^n \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(b,t)dt 
\]
hence
\[
< \phi_0, \Lambda(\phi_0) > = \int_0^T \left( \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(b,t) \right)^2 dt 
\]
which proves that \( \Lambda \) is an isomorphism and consequently the equation (14) has a unique solution which corresponds to the state gradient to be estimated on the subregion \( \omega \).

4.2 Zone sensor case

Here we consider the system (1) with the output function (4). For \( \phi_0 \) in \( \bar{F} \) the system (10) gives the solution \( \phi \) and we consider the semi-norm on \( \bar{F} \) defined by:
\[
\bar{\phi}_0 \rightarrow ||\phi_0||_\bar{F}^2 = \int_0^T \left( \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(t), f >_{L^2(D)} \right)^2 dt
\]
and the system:
\[
\left\{ \begin{array}{l}
\frac{\partial \psi_1}{\partial t}(x,t) = -A^* \psi_1(x,t) - v(t)f(x) \chi_D \quad Q \\
\frac{\partial \psi}{\partial \nu_{A^*}}(\xi,t) = 0 \\
\psi_1(x,T) = 0 \quad \Omega
\end{array} \right.
\]
where \( v(t) = \sum_{k=1}^n < \frac{\partial \phi}{\partial x_k}(t), f >_{L^2(D)} \).
We consider the operator
\[
\Lambda : \bar{F} \rightarrow \bar{F}^* \\
\phi_0 \rightarrow P(\Psi_1(0))
\]
and the system:
\[
\left\{ \begin{array}{l}
\frac{\partial \psi}{\partial t}(x,t) = -A^* \psi(x,t) - u(t)f(x) \chi_D \quad Q \\
\frac{\partial \psi}{\partial \nu_{A^*}}(\xi,t) = 0 \\
\psi(x,T) = 0 \quad \Omega
\end{array} \right.
\]
where \( u(t) = \sum_{k=1}^n < \frac{\partial \phi}{\partial x_k}(t), f >_{L^2(D)} \).

Then the regional gradient observability turns up to solve the equation:
\[
\Lambda(\phi_0) = P(\Psi(0))
\]
where \( \Psi = (\psi(0), ..., \psi(0)) \) and \( \psi \) is the solution of the system (17).

Then we have the following result:

**Proposition 4.2**

If the sensor \((D, f)\) is G-strategic on \( \omega \) then the equation (18) has a unique solution which is the gradient of the initial state to be observed on \( \omega \).

The proof is similar to the pointwise case.

5 Numerical Approach

We consider the system (1) observed by a pointwise sensor located at \( b \in \Omega \). In the previous section, it has been seen that the regional observability of the initial gradient on \( \omega \) turns up, in all cases, to solve the equation
\[
\Lambda(\phi_0) = P(\Psi(0))
\]
The solution of this equation may be achieved very easily when one can calculate the components \( \Lambda_{ij}^{kl} \) of \( \Lambda \) in a suitable basis \{ \( (\varphi_i); i = 1, ..., \infty, j = 1, ..., n \} \) of \( (L^2(\Omega))^n \). The problem will then be approximated by the solution of the linear system
\[
\sum_{j=1}^n \sum_{i=1}^M \Lambda_{ij}^{kl} \varphi_{0ij} = (P(\Psi(0)))_{kl}, \quad (20)
\]
where \( (P(\Psi(0)))_{kl} \) are the components of \( P(\Psi(0)) \) in the considered basis.

Let \( \{ \varphi_i \}_{i \in \mathbb{N}} \) be a complete set of eigenfunctions of the operator \( A \) in \( H^1(\Omega) \). We consider also a basis of \( L^2(\Omega) \) denoted by \( \varphi_i \). Then the components \( \Lambda_{ij}^{kl} \) are solutions of the equation
\[
\sum_{j=1}^n \sum_{i=1}^\infty < \frac{\partial \varphi_m}{\partial x_j}, \varphi_i > < \frac{\partial \varphi_p}{\partial x_i}, \varphi_k > \Lambda_{ij}^{kl} = e^{(\lambda_m + \lambda_p)T} - 1 \sum_{j=1}^n \frac{\partial \varphi_m}{\partial x_j} \varphi_k - \Lambda_{ij}^{kl} \]
\[
m, p = 1, ..., \infty
\]
Indeed, we have seen in the previous paragraph that
\[ < \Lambda \tilde{\varphi}_0, \tilde{\varphi}_0 > = \int_0^T \left( \sum_{j=1}^n \frac{\partial \phi}{\partial x_j} (b, t) \right)^2 dt \]
from \( \phi(t) = \sum_{m=1}^\infty e^{\lambda_m t} < \phi_0, \varphi_m > H^1(\Omega) \varphi_m \)
we have
\[ \sum_{j=1}^n \frac{\partial \phi}{\partial x_j} (b, t) = \sum_{j=1}^n \sum_{m=1}^\infty e^{\lambda_m t} < \phi_0, \varphi_m > H^1(\Omega) \frac{\partial \varphi_m}{\partial x_j} (b) \]
then
\[ \left\{ \begin{array}{l}
\int_0^T \left( \sum_{j=1}^n \frac{\partial \phi}{\partial x_j} (b, t) \right)^2 dt \\
= \sum_{j=1}^n \sum_{p=1}^\infty < \phi_0, \varphi_p > H^1(\Omega) < \phi_0, \varphi_m > H^1(\Omega) \\
\times \frac{\partial \varphi_p}{\partial x_j} (b) \frac{\partial \varphi_m}{\partial x_j} (b) e^{(\lambda_m + \lambda_p) T} - 1 \lambda_m + \lambda_p .
\end{array} \right. \] (22)
On the other hand since \( \frac{\partial \phi_0}{\partial x_j} \in L^2(\Omega) \) we can develop it in the basis \((\varphi_i)\), then
\[ \frac{\partial \phi_0}{\partial x_j} = \sum_{m=1}^\infty < \frac{\partial \phi_0}{\partial x_j}, \varphi_m > L^2(\Omega) \varphi_m \]
and \( \tilde{\phi}_0 = \sum_{j=1}^n \sum_{i=1}^\infty < \frac{\partial \phi_0}{\partial x_j}, \varphi_i > L^2(\Omega) \varphi_i \)
where \( \varphi_i = (0, ..., 0, \varphi_i, 0, ..., 0) (\varphi_i) \) is at \( j^{th} \) column, for \( 1 \leq j \leq n \) is a basis of \( (L^2(\Omega))^n \).
But \( \frac{\partial \phi_0}{\partial x_j} = \sum_{m=1}^\infty < \phi_0, \varphi_m > H^1(\Omega) \frac{\partial \varphi_m}{\partial x_j} \)
then
\[ \tilde{\phi}_0 = \sum_{j=1}^n \sum_{i, m=1}^\infty < \phi_0, \varphi_m > H^1(\Omega) < \frac{\partial \varphi_m}{\partial x_j}, \varphi_i > L^2(\Omega) \varphi_i \]
thus
\[ \left\{ \begin{array}{l}
< \Lambda \phi_0, \phi_0 > \\
= \sum_{j=1}^n \sum_{p=1}^\infty < \phi_0, \varphi_p > H^1(\Omega) < \phi_0, \varphi_m > H^1(\Omega) \\
( \sum_{i, k=1}^\infty < \frac{\partial \varphi_m}{\partial x_j}, \varphi_i > L^2(\Omega) < \frac{\partial \varphi_p}{\partial x_k}, \varphi_k > L^2(\Omega) \\
< \Lambda \varphi_i, \varphi_k > (L^2(\Omega))^n \end{array} \right. \] (23)
Let \( \Lambda_{ij}^{kl} = < \Lambda \varphi_i, \varphi_k > (L^2(\Omega))^n \) from (22) and (23), we obtain (21).}

**Remark**
In the case of zone sensor \((D, f)\) the same developments give
\[ \left\{ \begin{array}{l}
\sum_{j=1}^n \sum_{i, k=1}^\infty < \frac{\partial \varphi_m}{\partial x_j}, \varphi_i > < \frac{\partial \varphi_p}{\partial x_k}, \varphi_k > \Lambda_{ij}^{kl} \\
= e^{(\lambda_m + \lambda_p) T} - 1 \lambda_m + \lambda_p \sum_{j, l=1}^n < \frac{\partial \varphi_p}{\partial x_l}, f > L^2(D) \\
< \frac{\partial \varphi_m}{\partial x_j}, f > L^2(D) \end{array} \right. \]
for \( m, p = 1, ..., \infty \)

6 Simulations
In this section we develop a numerical example that leads to some conjectures related to the best sensor location, the results are related to the choice of the subregion and the estimated gradient to be observed.
Let \( \Omega = [0, 1] \) and consider the system
\[ \left\{ \begin{array}{l}
\frac{\partial y}{\partial t} (x, t) - 0.01 \frac{\partial^2 y}{\partial x^2} (x, t) = 0 \quad Q \\
y(0, t) = y(1, t) = 0 \quad ]0, T[ \end{array} \right. \] (24)
The output is given by means of a pointwise sensor \( z(t) = y(b, t) \), with \( b = 0.91, T = 2 \).
Let \( \omega = [0.15, 0.85] \) and \( g(x) = 2x(x-1)(2x-1) \) be the initial gradient to be observed on \( \omega \).
The simulations give :

![Figure 2: Real initial gradient g (dashed line) and the estimated gradient \( \tilde{g} \) (continuous line) on \( \omega \).](image)

The initial gradient \( \tilde{g} \) is estimated with a reconstruction error \( ||\tilde{g} - g||_{L^2(\omega)} = 1.398 \times 10^{-4} \)

6-a Relation estimated gradient error - pointwise sensor location
The following simulation results show the evolution of the
observed gradient error with respect to the sensor location (see fig. 3). It shows that:

- For a given subregion and an initial gradient, there is an optimal sensor location (optimal in the sense that it leads very close to the initial gradient).
- When the sensor is located sufficiently far from the subregion $\omega$, the gradient error is constant for any location.
- The worse observed locations correspond to non $G$-strategic sensors on $\Omega = [0, 1]$ where $b \in \{\frac{2k + 1}{2m} | k \in [0, m - 1] \cap \mathbb{N}; 1 \leq m \leq 5\}$ (the order of approximation of the system is five).

Figure 3: Evolution of the estimated gradient error with respect to the sensor location.

### 6-b Relation subregion area - estimated gradient error

The gradient reconstruction error depends on the area of the subregion. The following table shows that both the error and the subregion area increase or decrease. The G-observability is realized by means of one pointwise sensor located in $b = 0.91$. The results are similar for other types of sensors.

<table>
<thead>
<tr>
<th>Subregion $\omega$</th>
<th>$|g - \hat{g}|_{L^2(\omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0.1, 0.9]$</td>
<td>$3.890 \times 10^{-4}$</td>
</tr>
<tr>
<td>$[0.29, 0.72]$</td>
<td>$9.290 \times 10^{-4}$</td>
</tr>
<tr>
<td>$[0.32, 0.67]$</td>
<td>$7.231 \times 10^{-4}$</td>
</tr>
<tr>
<td>$[0.53, 0.53]$</td>
<td>$6.862 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 1: Relation between the reconstruction gradient error and the subregion area.

### 7 Conclusion

This paper concerns the study of the gradient observability concept which is motivated by many real applications where the objective is to obtain informations about the state gradient in a given subregion. Moreover we have explored an approach which allows the reconstruction of the gradient. The problem where the subregion $\omega$ is a part of boundary of the system domain is under consideration and the work will appear in a separate paper.

### References


