The theory of maximal output admissible set for nonlinear discrete systems and application to perturbed systems

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Abstract

Consider the discrete nonlinear system \( x(i + 1) = f(x(i)), i \geq 0 \) and the corresponding output signal \( y(i) = Cx(i), i \geq 0 \). Given a constraint set \( \Omega \subset \mathbb{R}^p \), a initial state \( x(0) \) is said to be output admissible if the resulting output function satisfies the condition \( y(i) \in \Omega, \forall i \geq 0 \). The set of all possible such initial conditions is the output maximal admissible set \( X_\infty \). Contrary to the linear case, the representation of the maximal output admissible set for nonlinear systems is certainly more complex and not available. However, we restrain in this paper to the theoretical and algorithmic characterization of the set \( X_\infty^M = X_\infty \cap B(0, M) \) where \( B(0, M) = \{ x \in \mathbb{R}^n / ||x|| \leq M \} \) with \( M \) is as large as we desire it. The maximal output admissible set for discrete delayed systems is also considered. As direct application of obtained results, we propose a technique that allows to determine, among all the perturbations susceptible to infect the initial state of a discrete nonlinear system, those which are relatively tolerable.

1 Introduction

Output admissible sets have important applications in the analysis and design of closed-loop systems with state and control constraints. Although, the theory of output admissible sets has been extensively treated (see[2-11]), in most of the works devoted to its study, the problem for nonlinear systems is not considered hence it’s applicability is severely limited.

The aim of this work is to present a contribution to the study of the maximal output admissible sets \( X_\infty \) for discrete nonlinear systems. More precisely, the objective is to characterize the initial conditions of an uncontrolled nonlinear discrete system whose resulting trajectory satisfies a specified pointwise-in-time constraint. Such problem have important applications, to illustrate that we consider the following example.

A nonlinear controlled discrete time system is described by

\[
\begin{cases}
  x(i + 1) = F(x(i)) + G(u_i), \quad i \in \mathbb{N} \\
  x(0) \quad \text{is given in } \mathbb{R}^n
\end{cases}
\]

where \( (u_i)_i \) is the feedback control given by

\[ u_i = H(x(i)) \]

\( F, G \) and \( H \) are supposed to be nonlinear appropriate functions. In addition, there may be physical constraints on the state variable if the constraints are violated by an action \( u_k \) serious consequences may happen, see ([1],[5]). By appropriate choice of matrices \( C \) and a set \( \Omega \), the constraints above may be summarized by the set inclusion

\[ Cx(i) \in \Omega, \quad \forall i \in \mathbb{N} \tag{1} \]

and it is desired to obtain a safe set of initial conditions \( x(0) \), i.e. a set \( X_\infty \) such that \( x(0) \in X_\infty \) implies (1), the problem can be state equivalently as a problem involving an unforced nonlinear discrete systems with output constraints. Specifically, given a continuous nonlinear function \( f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \) such that \( f(0) = 0 \), a \( p \times n \) matrix \( C \) and a constraint set \( \Omega \subset \mathbb{R}^p \).

We have to determine, for a given initial state \( x(0) \) if the system

\[
\begin{cases}
  x(i + 1) = f(x(i)), \quad i \in \mathbb{N} \\
  x(0) \quad \text{in } \mathbb{R}^n
\end{cases}
\]  

with the output signal

\[ y(i) = Cx(i), \quad i \in \mathbb{N} \tag{3} \]

satisfies the output constraint

\[ y(i) \in \Omega. \tag{4} \]

An initial condition \( x(0) \) is output admissible if the resulting output function (3) satisfied the constraint (4). The set of all such initial state is the maximal output admissible set \( X_\infty \).

In the case of a linear system (see [4], [12]) the maximal output set has completely determined, algorithm based on the mathematical programming, have allowed a numerical simulation of the maximal set. In the nonlinear case, that is the object of this paper, we have not been able to characterize
the set $X_{\infty}$; however, we propose a theoretical and algorithmic characterization of the set $X^M_{\infty} = X_{\infty} \cap B(0, M)$ where $B(0, M)$ is the ball of center 0 and ray $M$.

The fact to restrain to the set $X^M_{\infty}$ does not reduce the importance of the work and this for the next reason. Given an initial state $x(0) \in \mathbb{R}^n$, we wonders if $x(0)$ is output admissible or not. To answer to this question we firstly determines the set $X_{\infty} = X_{\infty} \cap B(0, r)$ where $r$ is a real that verifies $\|x(0)\| \leq r$ and in continuation we verifies if $x(0) \in X_{\infty}$ or no. The case of discrete delayed systems is also studied and simple examples are presented.

In the second part of this work, we consider the nonlinear disturbed discrete system described by

$$
\left\{ \begin{array}{l}
x^d(i+1) = f(x^d(i)), \quad i \in \mathcal{N} \\
x^d(0) = x(0) + d
\end{array} \right.
$$

(5)

where $(x^d(i))$ is the disturbed state of system, $d$ is a perturbation that infect the initial state $x(0)$. The corresponding output perturbation is supposed to be

$$y^d(i) = Cx^d(i).
$$

(6)

The perturbation $d$ being unavoidable, we use technique developed in the first part to determine all perturbations $d$ that are $\epsilon$-tolerable, i.e., the perturbations such that

$$\|y^d(i) - y(i)\| \leq \epsilon, \quad \forall i \geq 0
$$

where

$$y(i) = Cx(i), \quad i \geq 0
$$

and $(x(i))_{i \geq 0}$ is the uninfected state given by

$$
\left\{ \begin{array}{l}
x(i+1) = f(x(i)), \quad i \in \mathcal{N} \\
x(0) \in \mathbb{R}^n
\end{array} \right.
$$

(7)


2 Preliminary results

Consider the uncontrolled nonlinear discrete system described by

$$
\left\{ \begin{array}{l}
x(i+1) = f(x(i)), \quad i \in \mathcal{N} \\
x(0) \in \mathbb{R}^n
\end{array} \right.
$$

(7)

the corresponding output is

$$y(i) = Cx(i), \quad i \in \mathcal{N}
$$

(8)

where the state variable $x(i)$ is in $\mathbb{R}^n$, $f$ is a continuous nonlinear function that verify $f(0) = 0$ and the observation variable $y(i) \in \mathbb{R}^p$, satisfies the output constraint

$$y_i \in \Omega, \quad \forall i \in \mathcal{N}
$$

(9)

where $C$ is a $p \times n$ real matrix.

An initial condition $x(0) \in \mathbb{R}^n$ is output admissible if $x(0) \in B(0, M)$ and the resulting output function (8) satisfies (9). The set of all such initial conditions is the maximal output admissible set $X^M_{\infty}$.

We proof that under hypothesis on $f$, the maximal output admissible set is determined by a finite number of functional inequalities and leads to algorithmic procedures for the computation of $X^M_{\infty}$.

The system (8) can be equivalently rewritten in the form

$$y(i) = Cf(x_i), \quad \text{for all } i \in \mathcal{N}
$$

(10)

The set of all output admissible initial states is formally given by

$$X^M_{\infty} = \{ x_0 \in B(0, M) \cap \mathbb{R}^n / Cf(x_0) \in B(0, \epsilon), \quad \forall i \in \mathcal{N} \}
$$

(11)

We assume hereafter that $0 \in \text{int} \Omega$. This assumption is satisfied in any reasonable application and has nice consequences. Imposing special condition on $f$ we have a nonempty maximal admissible set $X^M_{\infty}$ which contains the origin, indeed

**Proposition 1** The set $X^M_{\infty}$ is closed and if $f$ is asymp-totically Liapunov stable (i.e., $\forall i \in \mathcal{N} \exists \delta > 0$ such that $\|x(I) - x'(I)\| < \delta$ then $\lim_{i \to \infty} \|f(x(I)) - f(x'(I))\| = 0$) and $0 \in \text{int} \Omega$ then, $0 \in \text{int} X^M_{\infty}$.

**Proof.**

It is easily to verify the closure of $X^M_{\infty}$ from his definition and continuity of $f$.

The assumption of Liapunov stability implies that there exists a constant $\delta > 0$ such that for all $x_0 \in B(0, \delta)$, $\lim_{i \to \infty} \|f(x_0)\| = 0$. which implies that for all $x_0 \in B(0, \delta)$ and $\eta > 0$ there exists $i_0 > 0$, such that for all $i \geq i_0$, $f(x_0) \in B(0, \eta)$, we deduce that

$$\forall x_0 \in B(0, \delta) \quad \text{we have } Cf(x_0) \in B(0, \eta\|C\|), \quad \forall i \geq i_0.
$$

(12)

Since $0 \in \text{int} \Omega$, $\exists \epsilon > 0$ such that $B(0, \epsilon) \subset \Omega$ then, if we pose $\eta = \frac{\epsilon}{\|C\|}$ then $\exists \delta > 0$ such that $x_0 \in B(0, \delta) \implies Cf(x_0) \in B(0, \epsilon) \subset \Omega$, $\forall i \geq i_0$ and using the continuity of $f$ and the condition $f(0) = 0$ we deduce that $\exists \delta' > 0$ such that $x_0 \in B(0, \delta') \implies Cf(x_0) \in \Omega$. $0 \leq i < i_0$ we choose $\alpha = \inf(\delta, \delta', M)$ we obtain,

$$y_i = Cf(x_0) \in \Omega, \quad 0 \leq i \leq i_0, \text{for every } x_0 \in B(0, \alpha), \text{thus } B(0, \alpha) \subset X^M_{\infty}, \text{consequently } 0 \in \text{Int} X^M_{\infty}.
$$

3 Characterization of the maximal output admissible set

In order to characterize the maximal output admissible set given formally by (11), we define for each integer $k$ the set

$$X^M_k = \{ x_0 \in B(0, M) \cap \mathbb{R}^n / Cf(x_0) \in \Omega, \quad \forall i \in \{0, \ldots, k\} \}
$$

**Definition 1** The set $X^M_k$ is finitely determined if there exists an integer $k$ such that $X^M_{\infty}$ is nonempty and $X^M_{\infty} = X^M_k$.

**Remark 1**

i) Obviously, the set $X^M_k$ satisfies the following condition:

for $k_1, k_2 \in \mathcal{N}$ such that $k_1 \leq k_2$, we have,

$$X^M_{\infty} \subset X^M_{k_2} \subset X^M_{k_1}$$
ii) Suppose that $X^M_k$ is finitely determined and let $k_0$ be the smallest $k$ such that $X^M_k = X^M_{k+1}$, then $X^M_\infty = X^M_{k_0}$ for all $k \geq k_0$.

**Proposition 2** $X^M_\infty$ is finitely determined if and only if there exists an integer $k$ such that $X^M_k$ is nonempty and $X^M_k = X^M_{k+1}$.

Proof. Suppose that it exists an integer $k$ such that $X^M_k$ is nonempty and $X^M_k = X^M_{k+1}$ then

$$x_0 \in X^M_k \implies x_0 \in X^M_{k+1} \implies f(x_0) \in X^M_{k+1}$$

and by iteration

$$x_0 \in X^M_k \implies f^j(x_0) \in X^M_k, \quad \forall j \in \mathbb{N}$$

hence $X^M_k \subset X^M_\infty$. We apply remark 1 to deduce that $X^M_k = X^M_\infty$.

Conversely, if $X^M_\infty = X^M_k$ for some $k \in \mathbb{N}$, then $X^M_k$ is nonempty and obviously $X^M_k = X^M_{k+1}$. ■

As a natural consequence of the previous proposition, we shall give in section 4 an algorithm which allows to determine the smallest integer $k^*$ such that $X^M_\infty = X^M_{k^*}$.

It is desirable to have simple condition which assure the finite determination of $X^M_\infty$. Our main results in this direction is the following theorems.

**Theorem 1** Suppose the following assumptions hold

1. $f$ is continuous, $f(0) = 0$ and $f$ is asymptotically Liapunov stable.
2. $0 \in \text{int}\Omega$.
3. $f(\lambda x) = g(\lambda)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda \in \mathbb{R}$ where $g$ is a real function which verify $g(0) = 0$ and the sequence $(g^k(\lambda))_{k \geq 0}$ is bounded $\forall \lambda \in \mathbb{R}$, $(g^k = g \circ g \circ \ldots \circ g)$. $k$-times

then $X^M_\infty$ is finitely accessible.

Proof. First case: $M \leq \delta$, then we apply equation (12) for $\eta = \frac{\delta}{\| Cf \|}$ we obtain

$$\| f(x) \| \leq \eta \| x \|, \quad \forall x \in \mathbb{R}^n \text{ and } \eta \in [0,1].$$

Second case: $M > \delta$, by hypothesis 3 of theorem we have $Cf^k(x_0) = Cg^k(M \frac{\delta}{\| Cf \|} x_0) = g^k(M \frac{\delta}{\| Cf \|})Cf^k(M \frac{\delta}{\| Cf \|} x_0)$. Since $(g^k(M \frac{\delta}{\| Cf \|}))_{k \geq 0}$ is bounded by some constant $M'$ then using equation (12) and $\eta = \frac{\delta}{\| Cf \| M'}$ we deduce that

$$\forall x_0 \in B(0,M) \exists i_0 \text{ such that } Cg^k(x_0) \in B(0,\epsilon), \quad \forall k \geq i_0.$$ In particular we obtain equation (13).

Then if $x_0 \in X^M_{i_0-1}$, we have $x_0 \in B(0,M)$ and $Cg^k(x_0) \in \Omega, \quad \forall k \in \{0, \ldots, i_0 - 1\}$, by equation (13) we deduce that $x_0 \in X^M_{i_0}$ Consequently $X^M_{i_0-1} = X^M_{i_0}$ and we deduce from proposition 1 that the maximal admissible set is finitely determined. ■

**Theorem 2** Suppose the following assumptions hold

1. $\| f(x) \| \leq \eta \| x \|, \quad \forall x \in \mathbb{R}^n \text{ and } \eta \in [0,1].$
2. $0 \in \text{int}\Omega$.

then $X^M_\infty$ is finitely accessible.

Proof. It apparent from hypothesis 1 that

$$\| Cg^k(x_0) \| \leq \| C \| \| \eta \| \| x_0 \|, \quad \forall i \in \mathbb{N},$$

then for $x_0 \in B(0,M)$ there exists a $k \in \mathbb{N}^*$ such that

$$\| Cg^k(x_0) \| \leq \epsilon \quad (14)$$

then, if $x_0 \in X^M_{k-1}$ we have $x_0 \in B(0,M)$ and $Cg^i(x_0) \in \Omega$ for $i \in \{0, \ldots, k-1\}$. Since $B(0,\epsilon) \subset \Omega$ and by equation (14) we have $Cg^i(x_0) \in \Omega$ for $i \in \{0, \ldots, k\}$. Consequently $x_0 \in X^M_k$. This result imply that $X^M_k = X^M_{k-1}$. ■

4 Algorithmic determination

The proposition 1 suggests the following conceptual algorithm for determining the output admissible index $k_0$, consequently the characterization of the set $X^M_\infty$.

**Algorithm I**

1. Set $k = 0$.
2. If $X^M_k = X^M_{k+1}$ then stop, else continue.
3. Replace $k$ by $k + 1$ and return to step 2.

Clearly, the algorithm I will produce $k_0$ and $X^M_k$ if and only if $X^M_\infty$ is finitely determined. There appears to be not finite algorithmic procedure for showing that $X^M_\infty$ is not finitely determined.

Algorithm I is not practical because it does not describe how the test $X^M_k = X^M_{k+1}$ is implemented. The difficulty can be overcome if we intruded in $\mathbb{R}^n$ the following norm

$$\| x \| = \max_{1 \leq i \leq n} | x_i |, \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

and if $\Omega$ is defined by:

$$\Omega = \{ y \in \mathbb{R}^n; \quad h_i(y) \leq 0, \quad i = 1, \ldots, s \} \quad (15)$$

where $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are a given function. Such a sets have many importance in a practical view. In this case, for every integer $k, X^M_k$ is given by

$$X^M_k = \{ x_0 \in B(0,M); \quad h_j(Cg^k(x_0)) \leq 0, \quad j = 1, \ldots, s; \quad i = 0, \ldots, k \}$$

On the other hand

$$X^M_{k+1} = \{ x_0 \in X^M_k; Cg^{k+1}(x_0) \in \Omega \} = \{ x_0 \in X^M_k; h_j(Cg^{k+1}(x_0)) \leq 0, \quad f o r j = 1, \ldots, s \}$$
Now, since \( X_k \subset X_k^M \) for every integer \( k \), then
\[
X_{k+1}^M = X_k^M \iff X_k^M \subset X_{k+1}^M
\]
\[
\iff x_0 \in X_k^M; \ h_j(C f^{k+1}(x_0)) \leq 0,
\quad \text{for all } j = 1, \ldots, s
\]
\[
\iff \sup_{x_0 \in X_k^M} h_j(C f^{k+1}(x_0)) \leq 0,
\quad \text{for all } j = 1, \ldots, s
\]
\[
\iff \sup_{h_j(C f^{k+1}(x_0)) \leq 0, \ g_r(x_0) \leq 0}
\quad \text{for all } j = 1, \ldots, s, \ l \in \{0, \ldots, k\},
\quad r \in \{1, \ldots, 2n\} \text{ and } i \in \{1, \ldots, s\}.
\]

Algorithm II

step 1: Let \( k = 0 \);
step 2: For \( i = 1, \ldots, s \), do:
Maximize \( J_i(x) = h_i(C f^{k+1}(x_0)) \)
\[
\begin{align*}
& h_j(C A^l x) \leq 0, \quad g_j(x) \leq 0 \\
& r = 1, \ldots, s, \quad j = 1, 2, \ldots, 2n, \\
& l = 1, \ldots, k.
\end{align*}
\]
Let \( J^* \) be the maximum value of \( J_i(x) \).
If \( J^* \leq 0 \), for \( i = 1, \ldots, s \), then
set \( k^* = k \) and stop.
Else continue.
step 3: Replace \( k \) by \( k + 1 \) and return to step 2.

Assumptions of theorems 1 and 2 are sufficient but not necessary. If these conditions are not verified, then Algorithm II can also be used for every \( \Omega \) given by (15). If the algorithm converges then \( X_\infty \) is finitely determined, else it is not. To illustrate this work we give some examples.

Example 1: Let \( f, C, \Omega, \) and \( M \) given by
\[
f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,
\quad \left( \begin{array}{c} x \\ y \end{array} \right) \rightarrow \left( \begin{array}{c} 0.4|x| - 0.1y \\ -0.2|x| + 0.5y \end{array} \right)
\]
\( C = (-1, 0.2), \ \Omega = [-0.5, 0.5] \) and \( M = 10 \).

Then, we use algorithm II to establish that \( k^* = 2 \) and we have
\[
X_\infty^M = \left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 : |x| \leq 10, |y| \leq 10, \\
-0.5 \leq -x + 0.2y \leq 5, \\
-0.5 \leq -0.44|x| + 0.11|y| \leq 5, \\
-0.5 \leq 0.176|x - 0.02y| - 0.04|x + 0.1y| \leq 5 \right\}
\]
The following figure gives a representation of Maximal output set \( X_\infty^M \) corresponding to example 1.

5 Maximal output admissible sets for non-linear discrete delayed Systems

Consider the uncontrolled nonlinear discrete delayed system described by
\[
\left\{ \begin{array}{l}
x(i+1) = f(x(i), \ldots, x(i-r)), \ i \in \mathbb{N} \\
x(0) = x_0 \\
x(s) = \alpha_x, \ -r \leq s \leq -1
\end{array} \right.
\]
the corresponding output is
\[
y(i) = \sum_{j=0}^{t} C_j x(i-j), \quad i \in \mathbb{N}
\]
where the state variable \( x(i) \) is in \( \mathbb{R}^n \), \( f \) is a continuous nonlinear function that verify \( f(0) = 0 \), and \( t \) are integers such that \( t \leq t \) and the observation variable \( y(i) \in \mathbb{R}^p \), satisfies the output constraint

\[
y(i) \in \Omega, \quad \forall i \in \mathbb{N}
\]

(18)

where \( C_j \) are a \( p \times n \) real matrices.

An initial condition \( \alpha = (x_0, \alpha_{-1}, \ldots, \alpha_{-r}) \in \mathbb{R}^{n(r+1)} \) is output admissible if the resulting output function (17) satisfies (18). The set of all such initial conditions is the maximal output admissible set \( X^M_\omega \).

In order to characterize the maximal output sets given formally by (19), we define for each integer \( k \) the set

\[
X^M_k = \{ \alpha \in B(0, M) \cap \mathbb{R}^{n(r+1)} / C \mathbb{F}^i(\alpha) \in \Omega, \quad \forall i \in \mathbb{N} \}.
\]

In order to characterize the maximal output sets given formally by (19), we define for each integer \( k \) the set

\[
X^M_k = \{ \alpha \in B(0, M) \cap \mathbb{R}^{n(r+1)} / C \mathbb{F}^i(\alpha) \in \Omega, \quad \forall i \in \{0, 1, \ldots, k\} \}.
\]

On the other hand

\[
F^i(\alpha) = (f(F^{i-1}(\alpha)), \ldots, f(F^{i-r-1}(\alpha)))^\top, \quad \forall i > r
\]

using equation (20), if \( f \) verify equation (21) with \( \|A_i(\lambda)\|_{i \geq 0} \) is bounded \( \forall \lambda \in \mathbb{R} \), \( \mathbb{F} \) is asymptotically Lyapunov stable and \( 0 \in \text{int} \Omega \) we can use results of theorem 1 of section 3 to characterize the set \( X^M_\omega \) described by equation (19).

On the other hand if

\[
\|F^i(x)\| \leq Nk^i\|x\|, \quad 0 < k < 1, \forall i \in \mathbb{N},
\]

we can use results of theorem 2 to characterize the output admissible set.

### 6 Application to Perturbed Systems

This section is devoted to the characterization of admissible disturbances for the nonlinear discrete infected system described by

\[
\begin{align*}
x^{d}(i+1) &= f(x^{d}(i)), \quad i \in \mathbb{N} \\
x^{d}(0) &= x_0 + d \in \mathbb{R}^n
\end{align*}
\]

the corresponding output function is

\[
y^{d}(i) = Cx^{d}(i), \quad i \in \mathbb{N}
\]

where \( f \) is a continuous nonlinear function, the observation variable \( y^{d}(i) \in \mathbb{R}^p \), and \( C \) is a \( p \times n \) real matrix, \( x^{d}(i) \in \mathbb{R}^n \) is the state variable and \( d \in \mathbb{R}^n \) represents an unavoidable disturbances which enters the system because of its connections with the environment. The output signal corresponding to \( d = 0 \) is simply denoted by \( (y(i))_{i \geq 0} \), i.e.,

\[
y(i) = Cx(i), \quad i \in \mathbb{N}
\]

where \( (x(i))_{i \geq 0} \) is the uninfected state given by

\[
\begin{align*}
x(i+1) &= f(x(i)), \quad i \in \mathbb{N} \\
x(0) &= x_0 \in \mathbb{R}^n
\end{align*}
\]

It is reasonable to agree that a source \( d \) is tolerable if, for every integer \( i \), the subsequent output variable \( y^{d}(i) \) remains in a neighborhood of the uninfected output \( y(i) \), this requires from us to introduce the index of admissibility \( \epsilon(\epsilon > 0) \). We say that the source \( d \) is \( \epsilon \)-admissible if \( g^{d}(i) - y(i) \| \leq \epsilon \) for all integer \( i \geq 0 \).

Motivated by practical consideration, we suppose that the disturbances \( d \) susceptible of infecting the initial state of system satisfy \( \varphi(d) = (x_0 + d, x_0) \in B(0, M) = \{ x \in \mathbb{R}^{2n} / \|x\| \leq M \} \).

The purpose of this section is to characterize, under certain hypothesis, the set \( S^M(\epsilon) \) of all source \( d \) such that
\( \varphi(d) \in B(0, M) \) which are \( \epsilon \)-admissible. We call \( S^M(\epsilon) \) the \( \epsilon \)-admissible set. We proof that under certain hypothesis on \( f \), the \( \epsilon \)-admissible set is finitely determined and leads to algorithmic procedures for the computation of \( S^M(\epsilon) \). The set of all \( \epsilon \)-admissible set is formally given by

\[
S^M(\epsilon) = \{ d \in \mathbb{R}^n / \varphi(d) \in B(0, M), \| C f^i(x_0 + d) - CF^i(x_0) \| \leq \epsilon, \ \forall i \in \mathbb{N} \}
\]

(26)

The set \( S^M(\epsilon) \) is derived from an infinite number of inequalities and it is difficult to characterize. However we propose some sufficient conditions which assure \( S^M(\epsilon) \) to be finitely accessible, i.e., there exists an integer \( k \) such that \( S^M(\epsilon) = S^M_k(\epsilon) \) where

\[
S^M_k(\epsilon) = \{ d \in \mathbb{R}^n / \varphi(d) \in B(0, M), \| C f^i(x_0 + d) - CF^i(x_0) \| \leq \epsilon, \ \forall i = 0, \ldots, k \}
\]

Let us define the functionals \( L \) and \( F \)

\[
L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
(a, b) \rightarrow a - b
\]

\[
F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n
\]

\[
(x, y) \rightarrow (f(x), f(y))
\]

by above notations we can easily establish that the set \( S^M(\epsilon) \) can be rewriting as follows

\[
S^M(\epsilon) = \{ d \in \mathbb{R}^n / \varphi(d) \in B(0, M), \| CLF^i\varphi(d) \| \leq \epsilon, \ \forall i \in \mathbb{N} \}
\]

Moreover

\[
S^M(\epsilon) = \{ d \in \mathbb{R}^n / \varphi(d) \in H^M(\epsilon) \}
\]

For every \( k \in \mathbb{N} \), we define the set \( H^M_k(\epsilon) \) by

\[
H^M_k(\epsilon) = \{ \xi \in B(0, M) / \| CLF^i\xi \| \leq \epsilon, \ \forall i \in \{0, \ldots, k\}, \}
\]

The proposition 3 we suggest the following conceptual algorithm for determining the output admissible index \( k_0 \), such that \( H^M_k(\epsilon) = H^M(\epsilon) \) and consequently the characterization of the set \( S^M(\epsilon) \) by

\[
S^M(\epsilon) = S^M_k(\epsilon) = \varphi^{-1}(H^M_k(\epsilon)).
\]

The set \( H^M_k(\epsilon) \) is described by

\[
H^M_k(\epsilon) = \left\{ \begin{array}{l}
\xi \in \mathbb{R}^{2n} / g_i(\xi) \leq 0 \\
h_j(CLF^i(\xi)) \leq 0
\end{array} \right\}
\]

with \( g_i : \mathbb{R}^{2n} \rightarrow \mathbb{R}, h_j : \mathbb{R}^{2p} \rightarrow \mathbb{R} \) are described for all \( x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \) and \( y = (y_1, \ldots, y_p) \) by

\[
\begin{aligned}
g_{2r-1}(x) &= x_r - M, \quad for \ r \in \{1, 2, \ldots, 2n\} \\
g_{2r}(x) &= -x_r - M, \quad for \ r \in \{1, 2, \ldots, 2n\}
\end{aligned}
\]

\[
\begin{aligned}
h_{2r-1}(y) &= y_r - \epsilon, \quad for \ r \in \{1, 2, \ldots, p\} \\
h_{2r}(y) &= -y_r - \epsilon, \quad for \ r \in \{1, 2, \ldots, p\}
\end{aligned}
\]

we deduce from

\[
H^M_k(\epsilon) = H^M_{k+1}(\epsilon) \implies H^M_k(\epsilon) \subset H^M_{k+1}(\epsilon)
\]

that \( H^M_k(\epsilon) = H^M_{k+1}(\epsilon) \implies \]

\[
\begin{cases}
\forall \xi \in H^M_k(\epsilon), & h_j(CLF^{k+1}(\xi)) \leq 0 \\
\forall j \in \{1, 2, \ldots, 2p\}
\end{cases}
\]

equivalently to \( \sup_{\xi \in H^M_k(\epsilon)} h_j(CLF^{k+1}(\xi)) \leq 0, \ \forall j \in \{1, 2, \ldots, 2p\} \). Consequently the test \( H^M_k(\epsilon) = H^M_{k+1}(\epsilon) \) leads to a set of mathematical programming problems, and we can give an practical algorithm under the following form.

**Theorem 3** Suppose the following assumptions hold

1. \( f \) is continuous and asymptotically Liapunov stable.
2. \( f(\lambda x) = g(\lambda)f(x), \ \forall x \in \mathbb{R}^n, \ \forall \lambda \in \mathbb{R} \) where \( g \) is a real function which verify \( g(0) = 0 \) and the sequence \( (g^k(\lambda))_{k \geq 0} \) is bounded for all \( \lambda \in \mathbb{R}, (g^k = g \circ g \circ \ldots \circ g) \)

then \( H^M(\epsilon) \) is finitely accessible.

**Theorem 4** If \( \| f(x) \| \leq \eta \| x \|, \ \forall x \in \mathbb{R}^n \) and \( \eta \in ]0, 1[, \) then \( H^M(\epsilon) \) is finitely accessible.
Algorithm III

step 1: Let $k = 0$;
step 2: For $i = 1, \ldots, 2p$, do:
Maximize $J_i(x) = h_i(CLF_{l+1}(x))$
\[ \begin{cases} h_i(CLF(x)) \leq 0, & g_i(x) \leq 0 \\ i = 1, \ldots, 2p, & j = 1, 2, \ldots, 4n, \ l = 1, \ldots, k. \end{cases} \]
Let $J_i^*$ be the maximum value of $J_i(x)$.
If $J_i^* \leq 0$, for $i = 1, \ldots, 2p$ then set $k^* := k$ and stop.
Else continue.
step 3: Replace $k$ by $k + 1$ and return to step 2.

To illustrate this work we give an example
Example 3: For
\[
\begin{align*}
\mathbf{f} & : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \\
\begin{pmatrix} x \\ y \end{pmatrix} & \mapsto \begin{pmatrix} \frac{1}{2} \sin \left(\frac{\pi}{2} \right) \\ \frac{1}{2} \sin \left(\frac{\pi}{2} \right) \end{pmatrix}
\end{align*}
\]
x_0 = 0, $C = (1,2)$, $\epsilon = 0.5$ and $M = 1$.
Then, we have $k^* = 1$ and
\[
X_\infty^M = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 / \| y \| \leq 1, \right. \\
\left. |x + 2y| \leq \frac{1}{2}, \\
\left\| \frac{1}{2} \sin \left(\frac{\pi}{2} \right) \right\| + y \leq \frac{1}{2} \right\}
\]
The following figure gives a representation of $\epsilon$-admissible set $S^M(\epsilon)$ corresponding to example 3.

Figure 3: The set $S^M(\epsilon)$ corresponding to example 3

References