Controllability Problem with a Minimum Energy for a Time-Varying Discrete Distributed System

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Abstract

The paper deals with the so-called extended exact controllability problem for time-varying discrete distributed systems. The purpose of this problem is to find the optimal control allowing the final state to be in a desired subset of the state space, with a minimum variable energy criterion. Using techniques analogous to the Hilbert Uniqueness Method, it is shown that the optimal control is characterized by a variational inequality in some appropriate Hilbert space.

1 Introduction

Controllability is considered as the most important concepts in the control theory. Contrarily to the finite dimensional case, one may distinguish several types of controllability for the infinite dimensional linear systems, as the exact controllability, approximate controllability, regional controllability, etc... These various topics have a great interest, especially when they are examined with a minimum energy criterion. The present paper deals with the so-called extended exact controllability problem for time-varying discrete distributed systems. This problem has first been introduced in [1] for hyperbolic systems; the results found were later generalized to continuous linear systems in [2].

Notations

- $l^2(\sigma^\beta_\alpha, Y)$ is a Hilbert space with usual addition, scalar multiplication and with an inner product defined by

$$\langle x, y \rangle_{l^2(\sigma^\beta_\alpha, Y)} = \sum_{i \in \sigma^\beta_\alpha} \langle x_i, y_i \rangle,$$

(1)

where $\sigma^\beta_\alpha$ denotes the finite family of $\mathbb{Z}$ defined by: $\sigma^\beta_\alpha = \{\alpha, \alpha + 1, \cdots, \beta - 1, \beta\}$. 

- For a family of operators $\{L_i : Y \to Y, \ i \in \sigma^\beta_\alpha, N \geq 1\}$, the operator $L$ is defined by:

$$L : l^2(\sigma^\beta_\alpha, Y) \to l^2(\sigma^\beta_\alpha, Y),$$

$$z \mapsto (L_0(z_0), \ldots, L_N(z_{N-1})).$$

- For a given finite sequence of operators $(A_k)_{k \in I}$ in a Hilbert space, the operator $\prod_{k=i}^j A_k$ denotes

$$\prod_{k=i}^j A_k = \begin{cases} A_i A_{i+1} \cdots A_j, & \text{if } i \leq j, \\ I & \text{if } j = i - 1, \\ 0 & \text{if } j < i - 1. \end{cases}$$

2 Statement of the problem

Consider the difference equation

$$\begin{cases} x_{i+1} = \Phi_i x_i + D_i v_i, & i \in \sigma^\beta_\alpha, \\ x_0 \in X \end{cases}$$

(2)

where $\{\Phi_i : X \to X, \ i \in \sigma^\beta_\alpha\}$ and $\{D_i : U \to X, \ i \in \sigma^\beta_\alpha\}$ are bounded.

Let $C$ be a non empty subset of $X$, closed, convex and bounded. The minimum energy control problem is to find the control law $v^* = (v_0, \ldots, v_{N-1})$ that satisfies

$$\begin{cases} x_N^v \in C \\ P(v^*) = \min \{P(u) : x_N^u \in C\} \end{cases}$$

where $x_N^v$ is the final state of system (2), corresponding to the control $u$; and $P$ is the energy criterion defined by

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\[ l^2(\sigma_0^{N-1}, U) \rightarrow \mathbb{R} : P(v) = \sum_{i=0}^{N-1} \langle R_i v_i, v_i \rangle, \quad (3) \]

such that \( R_i : U \rightarrow U \), for \( i \in \sigma_0^{N-1} \), are self-adjoint non-negative with \( \langle R_i u, u \rangle \geq \delta \| u \|^2 \), \( \delta > 0 \).

If \( \Sigma_C \) denotes the set of admissible controls

\[ \Sigma_C = \left\{ u \in l^2(\sigma_0^{N-1}, U) : x_N^u \in C \right\}, \]

the control problem \((\tilde{E})\) can be described by minimizing the criterion \( P \) over \( \Sigma_C \). Since \( C \) is closed and convex, it holds the same for \( \Sigma_C \), therefore the problem \((\tilde{E})\) admits a unique solution if and only if \( \Sigma_C \) is non-empty.

When \( C = \{ d \} \), the control problem \((\tilde{E})\) describes the standard exact controllability problem with a minimum energy. This case has been treated in [3] where the corresponding optimal control law is determined by introducing a convenient Hilbertian topology and an appropriate isomorphism. The same tools can be employed to resolve the problem \((\tilde{E})\) in its general version, (i.e \( C \) is any part of \( X \)). For this the following section summarizes some known results that are needed in the resolution of the considered problem.

### 3 Preliminary results

For an arbitrary control \( v \in l^2(\sigma_0^{N-1}, U) \), the corresponding final state can be given by

\[ x_N^v = \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + \sum_{j=0}^{N-1} \left( \prod_{k=0}^{N-j-2} \Phi_{N-k-1} \right) D_j v_j, \quad (4) \]

hence the controllability of system (2) is characterized as follows, (see [4]),

**Proposition 3.1** (i) The system (2) is exactly controllable if and only if

\[ \text{range } H = X. \]

(ii) The system (2) is approximately controllable if and only if

\[ \ker H^* = \{ 0 \}, \]

\( H \) is the controllability operator given by

\[ H : l^2(\sigma_0^{N-1}, U) \rightarrow X \]

\[ H(v) = \sum_{j=0}^{N-1} \left( \prod_{k=0}^{N-j-2} \Phi_{N-k-1} \right) D_j v_j; \quad (5) \]

and its adjoint is defined by

\[ H^* : X \rightarrow l^2(\sigma_0^{N-1}, U) \]

\[ H^* y = ((H^* y)_0, \ldots, (H^* y)_{N-1}), \]

with

\[ (H^* y)_j = D_j^* \left( \prod_{k=j+1}^{N-1} \Phi_k^* \right) y, \quad j \in \sigma_0^{N-1}. \quad (6) \]

Define over the space \( X \) the scalar product

\[ \langle x, y \rangle_{X_E} = \langle H_R^* x, H_R y \rangle_{l^2(\sigma_0^{N-1}, U)}, \quad (7) \]

where \( H_R \) is the control-energy operator defined by

\[ H_R = H \bar{R}^{-\frac{1}{2}}. \quad (8) \]

Assuming that the system (2) is approximately controllable; we deduce from the last proposition that the semi-norm \( \| \cdot \|_{X_E} \), obtained from the scalar product (7), given by

\[ \| y \|_{X_E} = \left\{ \sum_{j=0}^{N-1} \bar{R}^{-\frac{1}{2}} D_j^* \left( \prod_{k=j+1}^{N-1} \Phi_k^* \right) y \right\}^2, \quad (9) \]

defines a norm on \( X \). Therefore \( X \) is identified with a dense subspace of \( X_E \), where \( X_E \) is the completion space of \( X \) for the norm \( \| \cdot \|_{X_E} \).

In the other part, let \( \Lambda_R \) be the bounded selfadjoint operator defined on \( X \) by

\[ \Lambda_R : X \rightarrow X : \Lambda_R (x) = H_R H_R^* x, \quad (10) \]

so \( \Lambda_R \) has an extension to an isomorphism, denoted also by \( \Lambda_R \), defined from \( X_E \) to its dual \( X_E^* \).

### 4 Main result

The following theorem describes how the optimal solution of the problem \((\tilde{E})\) is characterized by a variational inequality established in \( X_E \), with the isomorphism \( \Lambda_R \).

**Theorem 4.1** Suppose that the system (2) is approximately controllable and \( \Sigma_C \neq \emptyset \).

The optimal control \( u^* = (u_0^*, \ldots, u_{N-1}^*) \) solution of the problem \((\tilde{E})\) is given by

\[ u_i^* = R_i^{1/2} D_i^* \left( \prod_{k=j+1}^{N-1} \Phi_k^* \right) f^*, \quad i \in \sigma_0^{N-1}, \quad (11) \]

where \( f \) is the unique element in \( X_E \), satisfying

\[ \langle f^*, x_N^v - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 - \Lambda_R (f^*) \rangle \geq 0, \quad \forall v \in \Sigma_C. \quad (12) \]
Moreover the optimal energy of the problem is
\[ P(u^*) = \|f^*\|^2_{X_E}. \] (13)

**Proof.** First let prove the uniqueness of \( f^* \). Suppose that there exists another element \( g \in X_E \) which satisfies the inequality (12), then for the control \( w = R^{-1}H^*(g) \), it holds
\[ \langle f^*, x_N^w - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 - \Lambda_R(f^*) \rangle \geq 0; \] (14)
since \( H(w) = \Lambda_R(g) \), we obtain
\[ \langle f^*, \Lambda_R(g) - \Lambda_R(f^*) \rangle \geq 0. \] (15)
By a same manner, we have also
\[ \langle g, \Lambda_R(f^*) - \Lambda_R(g) \rangle \geq 0, \] (16)
therefore
\[ \langle g - f^*, \Lambda_R(g) - \Lambda_R(f^*) \rangle = \|g - f^*\|^2_{X_E} \leq 0, \] (17)
thus \( g = f^* \).
To prove the existence of \( f^* \), let introduce the penalized minimizing problem described by
\[ \min_{C \times \mathbb{R}(\sigma_0^{N-1}, U)} P_\varepsilon(x, u), \] (18)
such that
\[ P_\varepsilon(x, v) = \frac{1}{\varepsilon} \|x^v_N - x\|^2_{X_E} + P(v), \varepsilon > 0; \] (19)
\( x^v_N \) is the final state corresponding to the control \( v \).
It is shown in [5] that for any \( \varepsilon > 0 \), the problem (18) has a solution \((x_\varepsilon, u_\varepsilon)\) characterized by
\[ u_\varepsilon = R^{-1}_0 D_\varepsilon \left( \prod_{j=1}^{N-1} \Phi_k \right) f_\varepsilon, \quad i \in \sigma_0^{N-1}, \] (20)
where \( f_\varepsilon \) is given by
\[ f_\varepsilon = \frac{1}{\varepsilon} (x_\varepsilon - x^v_N), \] (21)
and satisfies
\[ \langle f_\varepsilon, x - x_\varepsilon \rangle \geq 0, \forall x \in C. \] (22)
Furthermore, the sequence \( \langle (x_\varepsilon, f_\varepsilon, u_\varepsilon) \rangle \) is bounded in the space \( X \times X_E \times l^2(\sigma_0^{N-1}, U) \), so there exists an extracted subsequence of \((x_\varepsilon, f_\varepsilon, u_\varepsilon)\) which converges weakly to \((x^*, f^*, u^*)\) as \( \varepsilon \) is decreasing to zero. Combining this fact with the above relations, we show that \( x^* = x^v_N \), and the pair \((u^*, f^*)\) is given by formulae (11) and (12).

**Remark 4.1** It is known that the approximate controllability provides a norm over the state space \( X \). If this hypothesis fails, an analogous result to the above theorem may be established by considering the space \( X = X_0 \oplus X_1 \) where
\[ X_0 = \{ z \in X : \|z\|_{X_E} = 0 \}, \]
and
\[ X_1 = \{ h \in X : \langle h, z \rangle = 0, \quad \forall z \in X_0 \}. \]
Therefore the semi-norm \( \|\cdot\|_{X_E} \) defines a norm over \( X_1 \); and the inequality (12) is stated in \( \tilde{X}_E \), the completion space of \( X_1 \) for \( \|\cdot\|_{X_E} \).

**Remark 4.2** Notice that when \( C = \{ d \} \), the variational inequality (12) is reduced to a linear equation established in [3].

## 5 Closed ball case

For \( d \), a given state in \( X \), let look for the optimal control law with a minimum energy \( P \), driving the system (2) at the final stage \( N \), towards a desired closed neighborhood of \( d \). In this case the set of admissible controls is described by
\[ \Sigma^\alpha_d = \{ u \in l^2(\sigma_0^{N-1}, U) : \|x^u_N - d\| \leq \alpha \}; \]
\( \alpha > 0 \), is fixed.
The minimum energy controllability problem is to find \( v^* = (v_0^*, \ldots, v_{N-1}^*) \) satisfying
\[ \left( E^\alpha_d \right) \]
\[ \begin{cases} \|x^v_N - d\| \leq \alpha, \\ P(v^*) = \min_{v \in \Sigma_d} P(v) \end{cases} \]
It is clear that the problem \( (E^\alpha_d) \) coincides with \( \tilde{E} \) for \( C = B(d, \alpha) \), the closed ball centered at \( d \) with radius \( \alpha \).

**Proposition 5.1** For any \( \alpha > 0 \), \( (E^\alpha_d) \) has a unique solution if and only if
\[ d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 \in \text{range } H. \] (23)

**Proof.** It suffices to prove that the property (23) is equivalent to that \( \Sigma^\alpha_d \neq \emptyset \).
\[ d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 \in \text{range } H \]
\[ \iff \forall \alpha > 0, \text{there exists } y \in \text{range } H \text{ such that} \quad \left\| d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 - y \right\| \leq \alpha \]
Theorem 5.1 Suppose that system (2) is approximately controllable. For all \( \alpha \in \big(0, \infty\big) \), the optimal solution of problem \((E_a^\alpha)\) is given by

\[
v_i^* = R_i^{-1} D_i^* \left( \prod_{k=j+1}^{N-1} \Phi_k \right) f_a, \quad i = \sigma_0^{N-1},
\]

such that \((a, f_a) \in \mathbb{R} \times X_E\) is the unique pair satisfying

\[
\begin{cases}
\|a f_a\| = \alpha, \\
(aI + \Lambda_R) f_a = d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0.
\end{cases}
\]

Proof. First, notice that since the system (2) is approximately controllable we have

\[
d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 \in \text{range \( H \)}
\]

this assures a unique solution for \((E_a^\alpha)\). Now, the control \(v^* = (v_0^*, \ldots, v_{N-1}^*)\) may be written as

\[
v^* = R^{-1} (H^* f_a),
\]

hence

\[
x_N^v = \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + \Lambda_R(f_a);
\]

then, from equation (25) it holds

\[
\|x_N^v - d\| = \|a f_a\| = \alpha,
\]

which implies \(v^* \in \Sigma_a^\alpha\).

On the other hand, for \(v \in \Sigma_d^\alpha\), we have

\[
\langle x_N^v - \Lambda_R(f_a) - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0, f_a \rangle = \langle (x_N^v - d), f_a \rangle + a \|f_a\|^2
\]

since

\[
\langle (x_N^v - d), f_a \rangle \geq -\|x_N^v - d\| \cdot \|f_a\| \geq -a \|f_a\|,
\]

it follows that

\[
\langle x_N^v - \Lambda_R(f_a) - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0, f_a \rangle \geq 0
\]

\(\forall v \in \Sigma_d^\alpha\); hence \(f_a\) satisfies the variational inequality (12) of Theorem 2, and \(v^*\) is the optimal solution of the problem \((E_a^\alpha)\) such that \(a \|f_a\| = \alpha\). To prove the existence and uniqueness of \(a\), consider the functional \(S\) defined by \(S(a) = a \|f_a\|\); then we may show, as in [7], that the functional \(S\) is continuous, strictly increasing on \([0, +\infty[\) and verifies

\[
\begin{align*}
\lim_{a \to 0^+} S(a) &= 0, \\
\lim_{a \to +\infty} S(a) &= d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0,
\end{align*}
\]

therefore for any \(\alpha \in \big(0, \infty\big)\), \(d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0\) is unique such that

\[
S(a) = a \|f_a\| = \alpha.
\]

Remark 5.1 The assumption \(\alpha \in \big(0, \infty\big)\) is not at all restrictive, because if

\[
\left\| d - \left( \prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 \right\| \leq \alpha,
\]

then \(v^* = 0 \in \Sigma^\alpha_d\) is the optimal solution.

Remark 5.2 Theorem 3 provides us with a practical resolution of the controllability problem \((E_a^\alpha)\), it is based on the computation of the pair \((a, f_a)\) solution of the system (25); this can be realized by a dichotomy algorithm.
6 Conclusion

This paper have treated the so-called extended exact controllability problem for time-varying discrete distributed systems. It is shown that the optimal control is characterized by a variational inequality in some appropriate Hilbert space, which generalizes the exact controllability results (see [8], [5]). The results found may play a leading role on the linear quadratic control problem with final state in a convex subspace.

References


