Disturbance decoupling via static output feedback
for particular classes of linear systems

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Abstract

This work addresses the Disturbance Decoupling Problem in linear systems via static output feedback. Necessary and sufficient conditions for solvability in two important families of systems are established. The problem is solvable if and only if a given subspace verifies an invariance property. The set of output feedback matrices which solve the problem is then parameterized through a suitable change of the coordinate basis of the state, input and output spaces. A numerical example illustrates the proposed approach.

1 Introduction

The theoretical solution of the Disturbance Decoupling Problem with complete state feedback (DDP) is an excellent illustrative example of application of some of the fundamental concepts of the so-called geometric approach for linear control systems, such as \((A, B)\)-invariant and controllability subspaces. Geometric solvability conditions of DDP can be found in [1, 2].

The solution of DDP (when it exists) is not unique. It is an interesting problem to parameterize the set of all DDP controllers. This problem, for general systems is a very difficult one. Some works have however succeeded in doing this job for special types of systems. For instance, the complete solution and the parameterization of all DDP controllers for a particular class of linear systems, including the left-invertible systems, has been given in [3].

When the state vector is not available for measurement, one is led to consider the Disturbance Decoupling by Measurement Feedback (DDPM). This problem consists in finding a dynamic output feedback compensator such the transfer of the disturbances to the controlled output is zero. DDPM has been solved in [4, 5].

Little attention has been given to a problem of simpler formulation, the Disturbance Decoupling Problem by Static Output Feedback (DDPKM. The acronymous adopted in [6] is also adopted here). Its solvability conditions were firstly established in terms of the existence of a subspace with some special properties [7, 4]. However, the existence of this subspace is very difficult to check for in the general case.

A solution to DDPKM for left-invertible systems has been recently proposed in [6]. The system is firstly represented on a special coordinate basis in which DDPKM is solvable if and only if a certain submatrix is null.

In this paper, the results of [6] on DDPKM are extended by considering a broader set of systems. For two important classes of systems, geometric necessary and sufficient solvability conditions are established. Contrarily to [6], the conditions are stated in geometric terms, independently of the coordinate basis. A complete characterization of the solutions, as well as the parameterization of all corresponding output feedback matrices are also provided.

2 Preliminaries

Consider the linear, time-invariant, continuous-time system described by:

\[
\dot{x}(t) = Ax(t) + Bu(t) + Eq(t),
\]

\[
y(t) = Cx(t),
\]

\[
z(t) = Dx(t),
\]

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) is the control input, \(q \in \mathbb{R}^d\) is the disturbance input, supposed not directly available, \(y \in \mathbb{R}^l\) is the measured output and \(z \in \mathbb{R}^p\) is the output to
be controlled. Matrices $B$ and $E$ are supposed to have full column rank and matrices $C$ and $D$ are supposed to have full row rank.

The image spaces of matrices $B$ and $E$ will be denoted respectively by:

$$\mathcal{B} = \text{Im}(B), \quad \mathcal{E} = \text{Im}(E),$$

and the null spaces of matrices $C$ and $D$ will be denoted respectively by:

$$\mathcal{C} = \ker(C), \quad \mathcal{K} = \ker(D).$$

The goal of DDPKM is to determine, if possible, a static output feedback control law, $u(t) = Ky(t)$, such that the transfer matrix from disturbance $q$ to controlled output $z$ is null.

Two concepts are basic in the study of this problem, those of $(A, B)$-invariant and $(C, A)$-invariant subspaces. 

**Definition 2.1** Consider the pair $(A, B)$ related to the system (1). A subspace $\mathcal{V} \subset \mathbb{R}^n$ is said to be $(A, B)$-invariant if $A\mathcal{V} \subset \mathcal{V} + B$.

It can be shown that, equivalently, $\mathcal{V}$ is $(A, B)$-invariant if and only if there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.

It can also be shown that the set of all $(A, B)$-invariant subspaces contained in the subspace $\mathcal{K}$ has a supremum:

$$\mathcal{V}^* = \text{maximal } (A, B)\text{-invariant subspace contained in } \mathcal{K}.$$ 

**Definition 2.2** Consider the pair $(C, A)$ related to the system (1), (2). A subspace $\mathcal{S} \subset \mathbb{R}^n$ is said to be $(C, A)$-invariant if $A(\mathcal{S} \cap C) \subset \mathcal{S}$.

Equivalently, $\mathcal{S}$ is $(C, A)$-invariant if and only if there exists a matrix $G \in \mathbb{R}^{n \times l}$ such that $(A + GC)\mathcal{S} \subset \mathcal{S}$.

The set of all $(C, A)$-invariant subspaces containing the subspace $\mathcal{E}$ has an infimum:

$$\mathcal{S}_* = \text{minimal } (C, A)\text{-invariant subspace containing } \mathcal{E}.$$ 

The conditions for solvability of DDPM are given by the following theorem [7, 4]:

**Theorem 2.1** DDPKM is solvable if and only if there exists a subspace $\mathcal{V}$ which is both $(A, B)$- and $(C, A)$-invariant and such that $\mathcal{E} \subset \mathcal{V} \subset \mathcal{K}$.

Equivalently, DDPKM is solvable if and only if there is a subspace $\mathcal{V}$ and a matrix $K \in \mathbb{R}^{m \times l}$ such that $\mathcal{E} \subset \mathcal{V} \subset \mathcal{K}$ and $(A + BK)\mathcal{V} \subset \mathcal{V}$.

The preceding theorem establishes necessary and sufficient conditions for existence of a static output feedback which solves DDPKM, in terms of the existence of a subspace which verifies certain properties. However, it does not provide a constructive method to check for the existence of this subspace. This is a very difficult task in the general case.

As it will be seen in what follows, if the system verifies some conditions, then one can easily conclude about the existence of the aforementioned subspace, and, consequently, on the solvability of DDPKM. Furthermore, the characterization of the solutions and parameterization of all corresponding output feedback matrices are also readily obtained.

### 3 Solvability conditions for systems satisfying $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_*$

**Definition 3.1** [8, 2] An $(A, B)$-invariant subspace $\mathcal{V}$ contained in a subspace $\mathcal{K}$ is said to be self-bounded with respect to $\mathcal{K}$ if $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{V}$.

An $(A, B)$-invariant subspace $\mathcal{V} \subset \mathcal{K}$ is self-bounded with respect to $\mathcal{K}$ if for $x(0) \in \mathcal{V}$ the state vector cannot leave $\mathcal{V}$ through trajectories contained in $\mathcal{K}$.

A detailed characterization of self-bounded subspaces can be found in [8, 2]. Among the many properties of such subspaces, the following can be pointed out [2]:

**Property 3.1** The set of all self-bounded $(A, B)$-invariant subspaces contained in a subspace $\mathcal{K}$ is closed under intersection.

Consider now the following class of subspaces:

$$\mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*) \supseteq \{ \mathcal{V} : \mathcal{V} \in \mathcal{I}(A, B; \mathcal{K}) \text{ and } \mathcal{V} \supset \mathcal{S}_* \},$$

where $\mathcal{I}(A, B; \mathcal{K})$ represents the set of $(A, B)$-invariant subspaces contained in $\mathcal{K}$.

One should notice from Theorem 2.1 that a necessary condition for solvability of DDPKM is that the set $\mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*)$ be non-empty.

It will be assumed throughout this section that the considered systems satisfy the condition:

$$\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_*.$$ 

An important class of systems which verify this condition is that of systems whose transfer matrix of the triplet $(A, B, D)$, is left-invertible. Left-invertible systems are geometrically characterized by the condition [1, 2]: $\mathcal{B} \cap \mathcal{V}^* = \emptyset$.

In [6], DDPKM is solved uniquely for this type of system.

**Lemma 3.1** If $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_*$, then $\mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*)$ has an infimum:

$$\mathcal{V}_* = \min \mathcal{V} \in \mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*).$$

**Proof:** It suffices to show that $\mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*)$ is closed under intersection [1, 2]. For $\mathcal{V} \in \mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*)$, $\mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_* \subset \mathcal{V}$, hence $\mathcal{V}$ is a self-bounded $(A, B)$-invariant subspace and, from Property 3.1, the set $\mathcal{J}(A, B; \mathcal{K}, \mathcal{S}_*)$ is closed under intersection. □

The subspace $\mathcal{V}_*$ can be characterized in the following form:
Theorem 3.1 [2] Let $\mathcal{U}_c$ be the minimal $(D, A)$-invariant subspace containing $B$. Then $V_* = V^* \cap \mathcal{U}_c$.

Consider now the following set of state feedback matrices:

$$ F(V) \triangleq \{ F \in \mathbb{R}^{m \times n} : (A + BF)V \subset V \}. $$

Lemma 3.2

$$ F(V_*) \supset F(V) \forall V \in J(A, B; K, S_*). $$

Proof: It suffices to show that any matrix $F \in F(V)$, with $V \in J(A, B; K, S_*)$ also belongs to $F(V_*)$. By definition, $V_*$ is such that:

$$ AV_* \subset V_* + B, \ B \cap V^* \subset V_*, \ V_* \subset V. $$

Any $F \in F(V)$ is such that:

$$ (A + BF)V \subset V. $$

Hence,

$$ (A + BF)V_* \subset (A + BF)V \subset V. \quad (4) $$

Moreover, since $V_*$ is $(A, B)$-invariant and $BFV_* \subset B$,

$$ (A + BF)V_* \subset AV_* + B \subset V_* + B. \quad (5) $$

From relations (4), (5) above, one can conclude that:

$$ (A + BF)V_* \subset V \cap (V_* + B) \subset V_* + B \cap V \subset V_* + B \cap V^* \subset V_. $$

□

Based on the preceding results, the following solvability conditions for DDPKM, in the considered family of systems, can be established:

Theorem 3.2 For systems for which $B \cap V^* \subset S_*$, DDPKM is solvable if and only if $V_*$ is a $(C, A)$-invariant subspace.

Proof:

Necessity: If DDPKM is solvable, then there exist a matrix $K \in \mathbb{R}^{m \times l}$ and a subspace $V$ such that $S_* \subset V \subset V^*$ and $(A + BK)\bar{V} \subset V$. That guarantees that the set $J(A, B; K, S_*)$ is non-empty and, consequently, the existence of $V_*$. Moreover, by virtue of Lemma 3.2, $(A + BK)V_* \subset V_*$, hence $V_* = (C, A)$-invariant.

Sufficiency: If $V_*$ is $(C, A)$-invariant, since $V_* = (A, B)$-invariant as well and, by definition, $\mathcal{E} \subset V_* \subset K$, then from Theorem 2.1 DDPKM is solvable. □

It should be pointed out that the geometric conditions established by this Theorem are independent on the coordinate basis chosen to represent the state, input and output spaces. Moreover, unlike the conditions proposed in [6], they apply not only to left-invertible systems, but also to a much broader family.

Similar conditions are established in the following section for another important family of systems.

4 Solvability conditions for systems satisfying $V^* \subset C + S_*$

The geometric condition which defines the class of systems considered in this section is dual to that of the preceding section. The results presented in the sequel can also be easily obtained by duality. Therefore, in the aim of avoiding unnecessary repetition, the proofs will be omitted.

Definition 4.1 [2] A $(C, A)$-invariant subspace $S$ containing a subspace $\mathcal{E}$ is said to be self-hidden with respect to $\mathcal{E}$ if $S \subset S_* + C$.

A $(C, A)$-invariant subspace $S \supset \mathcal{E}$ is self-hidden with respect to $\mathcal{E}$ if it can become unobservable with a dynamic observer in the presence of disturbances belonging to $\mathcal{E}$.

An important property of self-hidden subspaces is now presented [2]:

Property 4.1 The set of all self-hidden $(C, A)$-invariant subspaces containing a subspace $\mathcal{E}$ is closed under subspace addition.

Consider now the following class of subspaces:

$$ \mathcal{J}(C, A; E, V^*) \triangleq \{ S : S \in I(C, A; E) \text{ and } S \subset V^* \}, $$

where $I(C, A; E)$ represents the set of $(C, A)$-invariant subspaces containing $\mathcal{E}$.

One should notice from Theorem 2.1 that a necessary condition for solvability of DDPKM is that the set $\mathcal{J}(C, A; E, V^*)$ be non-empty.

It will be assumed throughout this section that the considered systems satisfy the condition:

$$ V^* \subset C + S_*.$$

An important class of systems which verify this condition is that of systems whose transfer matrix of the triplet $(A, E, C)$, is right-invertible. Right-invertible $(A, E, C)$ triplets are geometrically characterized by the condition [2]: $C + S_* = \mathbb{R}^n$.

Lemma 4.1 If $V^* \subset C + S_*$, then $\mathcal{J}(C, A; E, V^*)$ has a supremum:

$$ S^* = \max S \in \mathcal{J}(C, A; E, V^*). $$

The subspace $S^*$ can be characterized in the following form:

Theorem 4.1 [2] Let $W^*$ be the maximal $(A, E)$-invariant subspace contained in $C$. Then $S^* = S_* + W^*$.

Necessary and sufficient conditions for solvability of DDPKM in the considered family of systems are given by the following Theorem:

Theorem 4.2 For systems for which $V^* \subset C + S_*$, DDPKM is solvable if and only if $S^*$ is an $(A, B)$-invariant subspace.
5 Parameterization of the solutions

From Theorems 3.1 and 4.1, one can notice that \( \mathcal{V}_s \) and \( \mathcal{S}^* \) can be obtained by computing \( \mathcal{V}^*, \mathcal{U}_s, \mathcal{S}_s \) and \( \mathcal{W}^* \), computation for which efficient and numerically stable algorithms are available \([9, 10]\). In particular, if one of the two conditions \( \mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_s \) or \( \mathcal{V}^* \subset \mathcal{C} + \mathcal{S}_s \) is verified, then it is always possible to determine orthogonal changes of basis such that the system can be represented by the following matrices:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (6)
\]

\[
E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}, \quad (7)
\]

\[
D = \begin{bmatrix} 0 & D_2 \end{bmatrix}, \quad (8)
\]

where \( B_{21} \) has full column rank, \( C_{11} \) has full row rank and

\[
\mathcal{V}_s = \text{Im} \left( \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \quad \text{(if } \mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_s \text{)}
\]

or

\[
\mathcal{S}^* = \text{Im} \left( \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \quad \text{(if } \mathcal{V}^* \subset \mathcal{C} + \mathcal{S}_s \text{)},
\]

where \( 0 \) and \( I \) represent respectively the null matrix and the identity matrix of appropriate dimensions.

One should notice that the structure of matrices \( D \) and \( E \) above are obtained assuming that the necessary condition for disturbance decoupling, \( DE = 0 \) \([1]\), is satisfied.

From now on, the first case (\( \mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_s \)) will be considered. For the second case (\( \mathcal{V}^* \subset \mathcal{C} + \mathcal{S}_s \)), similar results can be obtained by duality.

Under this assumption, the representation above reflects the following decomposition of the state, input and measured output spaces:

\[
\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \quad \mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2, \quad \mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2,
\]

where \( \mathcal{X}_1 = \mathcal{V}_s; \mathcal{B} \mathcal{U}_2 = \mathcal{B} \cap \mathcal{V}^* = \mathcal{B} \cap \mathcal{V}_s, \mathcal{B} \mathcal{U} = \mathcal{B}; \mathcal{C} \mathcal{X} = \mathcal{Y}, \mathcal{C}_2 \mathcal{X} = \mathcal{Y}_2, \ker(\mathcal{C}_2) = \mathcal{C} + \mathcal{V}_s.

Proposition 5.1 For systems for which \( \mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_s \), DDPKM is solvable if and only if there exists a matrix \( K_{11} \) such that:

\[
A_{21} + B_{22} K_{11} = 0. \quad (9)
\]

In this case, the set of all static output feedback matrices which solve DDPKM is given by:

\[
K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad (10)
\]

where \( K_{11} \) is given by the unique solution of the systems of linear equations (9) and \( K_{12}, K_{21} \) and \( K_{22} \) are freely assignable.

Proof: According to Theorem 3.2, it suffices to check if the \( (A, B) \)-invariant subspace \( \mathcal{V}_s = \text{Im} \left( \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \) is \( (C, A) \)-invariant as well, that is, if there exists a matrix \( K \) (10) such that \( (A + BK) \mathcal{V}_s \subset \mathcal{V}_s \). It can be verified by inspection that this is possible if and only if (9) is satisfied.

Moreover, from Lemma 3.2, one can conclude that the set of all matrix products \( KC \) which solve DDPKM is given by

\[
KC \in \mathbf{F}(\mathcal{V}_s). \quad \text{(if } 3 \text{)} \mathbf{F}(\mathcal{V}_s) = \{ F : A_{21} + B_{22} F_{11} = 0 \}. \quad \text{Hence, with } K \text{ given by (10) and by virtue of } C_{11} \text{ having full row rank, one can conclude that the submatrix } K_{11} \text{ must be the unique solution of } K_{11} C_{11} = F_{11} \text{ and consequently of equation (9) as well. The values of the other submatrices of } K \text{ are clearly irrelevant.} \quad \Box
\]

In the parameterization presented above, the degrees of freedom of the output feedback matrix become evident, allowing thereby their use in the satisfaction of other control goals besides disturbance decoupling.

One of such goals is the stabilization of the closed-loop system. One can easily verify that with this parameterization this problem is reduced to that of stabilizing via static output feedback the triplets \( (A_{11} + B_{11} K_{11} C_{11}, B_{12}, C_{11}) \) and \( (A_{22} + B_{22} K_{11} C_{112}, B_{22}, C_{22}) \). That turns out to be a still open problem. Several numerical techniques, based on sufficient conditions, are however available for its solution \([11]\). Some necessary geometric conditions can however be established.

The Disturbance Decoupling Problem by Static Measurement Feedback with Stability (DDPKMS) is solvable, for systems for which \( \mathcal{B} \cap \mathcal{V}^* \subset \mathcal{S}_s \), only if \( \mathcal{V}_s \) is both internally stabilizable and \( (C, A) \)-invariant externally stabilizable \([5, 2]\). Regarding the coordinate basis above, this amounts to state that there exist \( F_{21} \) and \( G_{22} \) such that all the eigenvalues of \( (A_{11} + B_{11} K_{11} C_{11} + B_{12} F_{21}) \) and \( (A_{22} + B_{22} K_{11} C_{12} + G_{22} C_{22}) \) respectively have negative real parts.

A sufficient condition is then the existence of \( K_{21} \) and \( K_{12} \) such that \( K_{21} C_{11} = F_{21} \) and \( B_{22} K_{12} = G_{22} \) respectively.

Once one succeeds in stabilizing the system, it is still possible to use the degrees of freedom left by DDPKM solution to optimize a quadratic performance index, as in e.g. \([10]\).

6 Numerical Example

Consider the system (1), (3) for which:

\[
A = \begin{bmatrix} -2 & 1 & -1 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 \\ 6 & -1 & 5 & -1 & 0 \\ 6 & -2 & 5 & -1 & 1 \\ 2 & 4 & 1 & 2 & 0 \end{bmatrix},
\]

\[
B = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & 1 \\ 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 0 \\ 4 & -2 \\ 5 & -2 \\ 2 & -2 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & 1 & -1 & 1 & 1 \end{bmatrix}.
\]
The solution of the disturbance decoupling problem via state feedback for this system was studied in [10, 3]. It can be checked that the system represented by these matrices satisfies the condition $\mathcal{B} \cap \mathcal{V}^* \subset S$.

Assume now that it is not possible to completely measure the state variables, but only a set of outputs, represented by equation (2), where:

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{bmatrix}.$$  

Applying the following changes of variables:

$$\bar{x} = Q^T x, \quad \bar{u} = Z_B^T u, \quad \bar{y} = Z_C^T y,$$

with the orthogonal matrices $Q, Z_B$ and $Z_C$ given by:

$$Q = \begin{bmatrix} 0.2700 & -0.3062 & -0.9065 & 0.1080 & 0 \\ 0.2400 & 0.4838 & -0.1700 & -0.6553 & 0.5 \\ -0.5100 & -0.1776 & -0.1700 & -0.6553 & -0.5 \\ -0.7800 & 0.1286 & -0.2455 & 0.2544 & 0.5 \\ 0.0300 & -0.7900 & 0.2455 & -0.2544 & 0.5 \end{bmatrix},$$

$$Z_B = \begin{bmatrix} -0.8944 & -0.4472 \\ -0.4472 & 0.8944 \end{bmatrix},$$

$$Z_C = \begin{bmatrix} -0.4352 & 0.1030 & 0.8944 \\ 0.8704 & -0.2061 & 0.4472 \\ 0.2304 & 0.9731 & 0 \end{bmatrix},$$

the matrices which represent the system take the form of equations (6)-(8):

$$\bar{A} = \begin{bmatrix} 0.8214 & 5.9177 & 8.1642 & 2.6890 & 4.6204 \\ 0.6105 & -1.0297 & 3.3197 & 2.7724 & -2.5933 \\ -0.1983 & 0.7245 & -0.4453 & 0.2105 & 1.3315 \\ -0.0439 & 0.1605 & 2.0476 & 0.9037 & 1.5303 \\ -0.1500 & 0.5481 & -0.2927 & -0.5281 & 0.7500 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 3.6897 & 0.6708 \\ 0.2111 & 0.5916 \\ -1.4780 & 0 \\ -0.3275 & 0 \\ -1.1180 & 0 \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} 3.6897 & 0.6708 \\ 0.2111 & 0.5916 \\ -1.4780 & 0 \\ -0.3275 & 0 \\ -1.1180 & 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 1.1720 & 0.3159 & -0.1301 & -0.9933 & 0.4480 \\ 0 & -0.3895 & -1.1725 & -1.9356 & 0.9216 \\ 0.0000 & 0.0000 & -0.1183 & -0.9930 & -0.4472 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

In this representation,

$$\bar{V}^* = \ker(\bar{D}) = \text{Im} \begin{bmatrix} I_4 \\ 0 \end{bmatrix},$$

$$\mathcal{B} \cap \bar{V}^* = \text{Im} \begin{bmatrix} 0.6708 \\ 0.5916 \end{bmatrix} \subset \text{Im} (\bar{E}) \subset \bar{V}^*,$$

$$\bar{V}_s = \text{Im} \begin{bmatrix} I_2 \\ 0 \end{bmatrix},$$

where $I_q$ represents the identity matrix of order $q$.

One can verify by inspection that $\bar{V}_s$ is $(C,A)$-invariant. Therefore, according to Theorem 3.2, DDPKM is solvable. From Proposition 5.1, the set of all output feedback matrices which solve DDPKM is parameterized by:

$$\bar{K} = \begin{bmatrix} -0.1145 & -1.3516 & X \\ X & X \end{bmatrix},$$

where the “X” elements are freely assignable.

It turns out that for this example $\bar{V}_s$ is both internally and externally stabilizable. Moreover, the triplets $(A_{11} + B_{11}K_{11}C_{11}, B_{12}, C_{11})$ and $(A_{22} + B_{21}K_{11}C_{12}, B_{21}, C_{22})$ are stabilizable via static output feedback. A DDPKMS solution is given by:

$$\bar{K} = \begin{bmatrix} -0.1145 & -1.3516 & -1.0 \\ -1.3 & -1.6 & 0 \end{bmatrix}.$$

### 7 Conclusion

Little attention had been given until a few years ago to the study of disturbance decoupling via static output feedback in linear systems. This is certainly due to the difficulty in establishing solvability conditions. It has been shown in this work that if the system satisfies certain properties, then it is possible not only to conclude about solvability, but also to parameterize the set of associated feedback matrices. Geometric necessary and sufficient conditions have then been established. Important classes of systems verify the aforementioned properties, among which left and right-invertible systems. This work thereby extends some results recently published, which are restricted to left-invertible systems, and whose solvability conditions are established with respect to a particular coordinate basis.

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