Pointwise control of the heat equation coupled with the linearized Navier-Stokes equations

P.-A. Nguyen, J.-P. Raymond
Laboratoire Mathématiques pour l’Industrie et la Physique
Université Paul Sabatier
31062 Toulouse Cedex 4 - France
{nguyen,raymond}@mip.ups-tlse.fr

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Abstract

We are interested in control problems for the linearized Navier-Stokes equations coupled with the heat equation in dimension $N \geq 2$. The control is a heat source localized on points. We establish the existence of optimal controls and optimality conditions.

1 Introduction

This paper deals with control problems for the system of the heat equation coupled with the linearized Navier-Stokes equations, in dimension $N \geq 2$. The control is localized on a point $b \in \Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a regular boundary $\Gamma$. The control variable $u$ belongs to some subset $K_U$ of $L^2(0,T)$. We consider the system $(S)$ defined by

$$\tag{S} \begin{cases} \frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div} (z \tau) = u \delta_b - y \cdot \nabla \theta \text{ in } Q, \\
\frac{\partial \tau}{\partial n} = 0 \text{ on } \Sigma, \quad \tau(0) = \tau_0 \in \Omega, \\
\frac{\partial y}{\partial t} - \Delta y + \sum_i \partial_i(z_i y_i) + \sum_i \partial_i(z y_i) + \nabla q = \beta \tau, \\
y(0) = y_0 \text{ in } \Omega, \\
\end{cases}$$

where $\kappa > 0$, $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$, $\theta$ belongs to $L^\infty(0,T;W^{1,\alpha}(\Omega))$, $z$ belongs to $L^\infty(0,T;L^\infty(\Omega))$, $\nabla q = 0$, $z \cdot n = 0$ on $\Sigma$, $\tau_0 \in \Omega$, $y_0 \in \Omega$, $\beta$ belongs to $L^\infty$, $\Sigma$, $y(0) = y_0 \in \Omega$.

The assumptions on $(\tilde{\alpha}, \alpha), (\tilde{m}, m)$ and $\rho$ are given below.

We study the optimal control problem

$$\begin{aligned}
&\text{inf} \{ J(\tau,y) \mid (\tau,y,u) \text{ satisfies } (S), u \in K_U \},
\end{aligned}$$

where

$$J(\tau,y) = \int_\Omega |y(T) - y_d|^p \, dx + \int_\Omega |\tau(T) - \tau_d|^p \, dx,$$

where $s > 1$. In the system $(S)$, $y$ denotes the fluid velocity, $q$ is the pressure, the temperature $\tau$ of the fluid is controlled by the heat source $u$, which is applied at the point $b$. The problem $(P)$ consists in minimizing the distance to some given profile $(\tau_d, y_d)$. Pointwise control of the one-dimensional Burgers equation is studied in [1], [2], [4]. Control problems for advection-diffusion equations are considered in [1], [2], [10]. In our knowledge the pointwise control of the heat equation coupled with the linearized Navier-Stokes equation has not been studied in the literature. Existence and regularity results for the system $(S)$ are established in Section 3. We mainly use the techniques introduced in [10] for convection-diffusion equations. Before studying $(S)$, we first transform the linearized Navier-Stokes equation by introducing the so-called Helmholtz decomposition.

For $1 < \ell < \infty$, we set

$$D\ell(\Omega) := \{ y \in D(\Omega; R^N), \text{ div } y = 0 \},$$

$$X\ell(\Omega) := D\ell(\Omega) \cap L^\infty(\Omega, R^N).$$

We denote by $G\ell(\Omega)$ the space of all vectors $\nabla p$ with $p \in W^{1,\ell}(\Omega)$. We have [6]

$$L^\ell(\Omega; R^N) = X\ell(\Omega) \oplus G\ell(\Omega).$$

The projection of $L^\ell(\Omega; R^N)$ into $X\ell(\Omega)$ is denoted by $P\ell ((I - P\ell)$ is the projection into $G\ell(\Omega)$). We set:

$$A\ell := P\ell \Delta,$$

with

$$D(A\ell) := W^{2,\ell}(\Omega; R^N) \cap W^{1,\ell}(\Omega; R^N) \cap X\ell(\Omega).$$

Then the closed operator $A\ell$ is the generator of an analytic semigroup $S\ell(t)_{t \geq 0}$ in $X\ell(\Omega)$ [6]. Since we look for a solution $y$ to system $(S)$ belonging to $L^\ell(0,T; X\ell(\Omega))$ for some $\ell > 1$, $r > 1$, the system $(S)$ can be written in the form:

$$\tag{S} \begin{cases} \frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div} (z \tau) = u \delta_b - y \cdot \nabla \theta \text{ in } Q, \\
\frac{\partial \tau}{\partial n} = 0 \text{ on } \Sigma, \quad \tau(0) = \tau_0 \in \Omega, \\
y' - A\ell y + P\ell \left( \sum_i \partial_i(z_i y_i) \right) + P\ell \left( \sum_i \partial_i(z y_i) \right) = P\ell(\beta \tau), \\
y(0) = P\ell(y_0), \\
\end{cases}$$

where $\kappa > 0$, $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$, $\theta$ belongs to $L^\infty$, $\Sigma$, $\tau_0 \in \Omega$, $y_0 \in \Omega$, $\beta$ belongs to $L^\infty$, $\Sigma$. The assumptions on $(\tilde{\alpha}, \alpha), (\tilde{m}, m)$ and $\rho$ are given below.
Throughout the paper, we make the following assumptions on \((q, m, \alpha, \rho)\).
\[
\begin{align*}
q > 2, \quad \infty > \tilde{m} > 2, \quad \frac{1}{\tilde{m}} + \frac{N}{2n} & \leq \frac{1}{2}, \\
\infty > \tilde{\alpha} > 2, \quad \alpha > \frac{Nq'}{2} \quad \frac{1}{\tilde{\alpha}} + \frac{N}{2\tilde{\alpha}} & \leq 1, \\
\rho > \max(m', \alpha'), \quad \rho & \geq \frac{N}{N - \frac{N}{\tilde{\alpha}}} \quad (3)
\end{align*}
\]

We also assume that the given profile \(\tau_d\) belongs to \(L^4(\Omega)\) and \(y_d\) belongs to \(X^q(\Omega)\).

2 Regularity results for the convection-diffusion equation

First we recall some regularity results needed in the sequel.

**Proposition 2.1** Let \(f\) be a function belonging to \(L^k(0,T; L^m(\Omega))\) with \(k > 1\) and \(k > 1\), and \(u\) belong to \(L^1(0,T)\). Let \(\theta_0\) belong to \(L^1(\Omega)\), \(z\) belong to \(L^\infty(0,T; (L^m(\Omega))^N)\) with \(\frac{1}{m} + \frac{N}{2m} \leq \frac{1}{2}\). The equation
\[
\begin{align*}
\frac{\partial \theta}{\partial t} - \Delta \theta + \text{div}(z \theta) &= f + u \theta_0 \quad \text{in } Q, \\
\frac{\partial \theta}{\partial n} &= 0 \quad \text{on } \Sigma, \quad \theta(0) = \theta_0 \quad \text{in } \Omega,
\end{align*}
\]

admits a unique solution \(\theta\) in \(L^r(0,T; L^p(\Omega)) + L^q(0,T; L^p(\Omega)) + L^\infty(0,T; L^p(\Omega))\) for all \((\tilde{r}, r, \tilde{p}, p)\) satisfying
\[
\begin{align*}
\tilde{r} > \tilde{k}, \quad \tilde{r} > \tilde{m'}, \quad r \geq k, \quad r > m', \quad \frac{1}{k} + \frac{N}{2k} & \leq \frac{1}{\tilde{r}} + \frac{N}{2r} + 1, \\
\tilde{r}_2 \geq q, \quad \tilde{r}_2 > \tilde{m'}, \quad r_2 > m', \quad \frac{1}{q} + \frac{N}{2q} & \leq \frac{1}{\tilde{r}_2} + \frac{N}{2r_2} + 1.
\end{align*}
\]

The mapping that associates \(\theta\) with \((f, u, \theta_0)\) is continuous from \((L^k(0,T; L^m(\Omega)) \times L^1(0,T) \times L^\infty(\Omega))\) into \((L^r(0,T; L^p(\Omega)) + L^q(0,T; L^p(\Omega)) + L^\infty(0,T; L^p(\Omega)))\). Moreover, \(\theta\) belongs to \(C([0,T]; L^\infty(\Omega)) + C([0,T]; L^\infty(\Omega))\) for all \((\nu, \nu_2)\) such that
\[
\begin{align*}
\nu > m', \quad \nu \geq k, \quad \frac{1}{k} + \frac{N}{2k} & \leq \frac{N}{2\nu} + 1, \\
\nu_2 > m', \quad \frac{1}{q} + \frac{N}{2q} & \leq \frac{N}{2\nu_2} + 1,
\end{align*}
\]

(where \(L^\infty(\Omega)\) denotes the space \(L^\infty(\Omega)\) endowed with its weak topology).

The mapping that associates \(\theta\) with \((f, u, \theta_0)\) is continuous from \(L^k(0,T; L^m(\Omega)) \times L^1(0,T) \times L^\infty(\Omega)\) into \((L^\infty(0,T; L^p(\Omega)) + L^\infty(0,T; L^p(\Omega)) + L^\infty(0,T; L^p(\Omega)))\).

**Proof.** This result can be proved by a argument similar to those in Propositions 2.6 and 2.7 [10]. \(\square\)

**Proposition 2.2** Let \(h\) be a field belonging to \(L^k(0,T; X^q(\Omega))\) with \(k > 1\) and \(k > 1\). Let \(y_d\) belong to \(X^q(\Omega)\), \(z\) belong to \(L^\infty(0,T; (L^m(\Omega))^N)\) with \(\frac{1}{m} + \frac{N}{2m} \leq \frac{1}{2}\). The equation
\[
\begin{align*}
y' &= -A_r y + P_r \left( \sum_i \partial_i (z_i y) \right) + P_r \left( \sum_i \partial_i (z_i y) \right) \\
\quad & \in P_r(h), \quad y(0) = P_r(y_0),
\end{align*}
\]

admits a unique solution \(y\) in \(L^r(0,T; X^p(\Omega)) + L^\infty(0,T; X^p(\Omega))\) for all \((\tilde{r}, r, \tilde{p}, p)\) satisfying
\[
\begin{align*}
\tilde{r} > \tilde{k}, \quad \tilde{r} > \tilde{m'}, \quad r \geq k, \quad r > m', \quad \frac{1}{k} + \frac{N}{2k} & \leq \frac{1}{\tilde{r}} + \frac{N}{2r} + 1.
\end{align*}
\]

The mapping that associates \(y\) with \((h, y_0)\) is continuous from \((L^k(0,T; X^q(\Omega)) \times X^q(\Omega))\) into \((L^r(0,T; X^p(\Omega)) + L^\infty(0,T; X^p(\Omega)))\).

Moreover, \(y\) belongs to \(C([0,T]; X^q(\Omega)) + C([0,T]; X^q(\Omega))\) for all \(\nu, \nu_2\) such that
\[
\begin{align*}
\nu > m', \quad \nu \geq k, \quad \frac{1}{k} + \frac{N}{2k} & \leq \frac{N}{2\nu} + 1,
\end{align*}
\]

The mapping that associates \(y\) with \((h, y_0)\) is continuous from \((L^k(0,T; X^q(\Omega)) \times X^q(\Omega))\) into \((L^r(0,T; X^p(\Omega)) + L^\infty(0,T; X^p(\Omega)))\).

**Proof.** This result can be proved, using the same argument as in the proof of Theorem 3.3 [10]. \(\square\)
3 State System

**Theorem 3.1** Under the assumptions (1)-(3), the state system (S) admits a unique solution \((\tau, y)\) in \((L^p(0, T; L^r(\Omega))) \times L^r(0, T; X^r(\Omega)))\), for all \((\bar{r}, r)\) satisfying:

\[
\bar{r} > \max(n', \bar{a}'d), \quad \bar{r} \geq q, \quad \rho \geq r > \max(m', \alpha'),
\]

\[
\frac{1}{q} + \frac{N}{2} < \frac{1}{\bar{r}} + \frac{N}{2r' + 1}.
\]

Moreover, the mapping that associates \((\tau, y)\) with \((u, \tau_0, y_0)\) is continuous from \(L^p(0, T) \times L^r(\Omega) \times X^r(\Omega)\) into \((L^p(0, T; L^r(\Omega))) \times L^r(0, T; X^r(\Omega)))\). Consequently, \(\tau\) belongs to \(C([0, T]; L^r(\Omega))\), and \(y\) belongs to \(C([0, T]; X^r(\Omega))\) for all \(u < \frac{N}{\bar{r}}\).

The mapping that associates \((\tau, y)\) with \((u, \tau_0, y_0)\) is continuous from \(L^p(0, T) \times L^r(\Omega) \times X^r(\Omega)\) into \((L^\infty(0, T; L^r(\Omega))) \times L^r(0, T; X^r(\Omega)))\).

**Proof.** 1 - Following the method in [10], we first prove the existence of a local solution. Let us set \(Q_T := \Omega \times [0, \bar{T}], \Sigma_T := \Gamma \times [0, \bar{T}]\). Let \((\bar{r}, r)\) be a pair obeying (4)-(5). By a fixed point method, we prove that the system

\[
\begin{align*}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div}(z \tau) &= u \partial_0 - y \cdot \nabla \theta \text{ in } Q_T, \\
\frac{\partial \tau}{\partial n} &= 0 \text{ on } \Sigma_T, \quad \tau(0) = \tau_0 \text{ in } \Omega, \\
y' - A_r y + P_r \left( \sum_i \partial_i (z_i y) \right) &= P_r(\beta \xi), \quad y(0) = P_r(y_0),
\end{align*}
\]

admits a solution \(\bar{\tau}\) small enough. Let \(\xi\) belong to \(L^p(0, \bar{T}; L^r(\Omega))\), and \((\xi, y)\) be the solution to the system:

\[
\begin{align*}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div}(z \tau) &= u \partial_0 - y \cdot \nabla \theta \text{ in } Q_T, \\
\frac{\partial \tau}{\partial n} &= 0 \text{ on } \Sigma_T, \quad \tau(0) = \tau_0 \text{ in } \Omega, \\
y' - A_r y + P_r \left( \sum_i \partial_i (z_i y) \right) &= P_r(\beta \xi), \quad y(0) = P_r(y_0),
\end{align*}
\]

Due to Proposition 2.2, from hypothesis (4)-(5) on \((\bar{r}, r)\), the unique solution \(y_\xi\) of (2) belongs to \(L^p(0, \bar{T}; L^r(\Omega))\). Moreover, we have \(\bar{r} > \bar{r}'\), and \(r > \alpha'\). Then \(y_\xi \cdot \nabla \theta \in L^\infty(0, \bar{T}; L^\infty(\Omega))\), where \(\frac{\bar{r}}{\bar{r}'} > 1\) and \(\frac{\bar{r}}{\alpha'} > 1\). From \(\frac{1}{\bar{r}} + \frac{N}{\bar{r}} < 1\), we use Proposition 2.1, which follows that the unique solution \(\xi_2\) belongs to \(L^p(0, \bar{T}; L^r(\Omega))\). Let \(\xi_1\) and \(\xi_2\) belong to \(L^p(0, \bar{T}; L^r(\Omega))\). Let \((y_{\xi_1}, z_{\xi_1})\) and \((y_{\xi_2}, z_{\xi_2})\) be the solutions to the system (s) corresponding to \(\xi_1\) and \(\xi_2\). Still with Proposition 2.1, we have:

\[
\begin{align*}
\|\tau_{\xi_1} - \tau_{\xi_2}\|_{L^r(0, \bar{T}; L^r(\Omega))} &\leq C\|\theta\|_{L^p(0, \bar{T}; L^r(\Omega))} \|y_{\xi_1} - y_{\xi_2}\|_{L^r(0, \bar{T}; L^r(\Omega))} \\
&\leq C\|\theta\|_{L^p(0, \bar{T}; L^r(\Omega))} \|z_{\xi_1} - z_{\xi_2}\|_{L^r(0, \bar{T}; L^r(\Omega))} \\
&\leq C\|\theta\|_{L^p(0, \bar{T}; L^r(\Omega))} \|\xi_1 - \xi_2\|_{L^r(0, \bar{T}; L^r(\Omega))},
\end{align*}
\]

where \(C_1\) can be chosen depending on \(T\), but independent of \(\bar{T}\). The mapping \(t \mapsto C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\) is absolutely continuous, then there exists \(\bar{T} > 0\) such that \(C_1(t_0)\|\theta(x, \tau)\|_{W^1,\infty(\Omega)} dt\).
then
\[ \|\tau_n\|_{L^1(0,T;\mathbb{R}^+) \cap L^\infty(0,T;\mathbb{R}^+)} \leq K((\|u\|_{L^1(0,T;\mathbb{R}^+)} + \|\tau_0\|_{L^1(0,T;\mathbb{R}^+)}) + \|\tau_0\|_{L^\infty(0,T;\mathbb{R}^+) + \|\nu\|_{X^*(\Omega)})). \]

Using estimate (6) for \( \hat{\tau} \), we obtain:
\[ \|\tau\|_{L^1(0,T;\mathbb{R}^+) \cap L^\infty(0,T;\mathbb{R}^+)} \leq \hat{K}((\|u\|_{L^1(0,T;\mathbb{R}^+)} + \|\tau_0\|_{L^1(0,T;\mathbb{R}^+) + \|\nu\|_{X^*(\Omega)}) \],

where \( \hat{K} \) is a constant depending on \( \|\beta\|_{L^\infty(0,T;L^\infty(\Omega))} \), on \( \|z\|_{L^\infty(0,T;L^\infty(\Omega))} \), and on \( \|\theta\|_{L^\infty(0,T;L^2(\Omega))} \).

4 - Estimate of the global solution. By induction, it is easy to prove that
\[ \|\tau\|_{L^1(0,T;\mathbb{R}^+) \cap L^\infty(0,T;\mathbb{R}^+)} + \|\nu\|_{X^*(\Omega)} \]
\[ \leq \hat{C}((\|u\|_{L^1(0,T;\mathbb{R}^+) + \|\tau_0\|_{L^1(0,T;\mathbb{R}^+) + \|\nu\|_{X^*(\Omega)})} \]

Due to Proposition 2.1, the state \((x, y)\) belongs to \( C([0, \infty]; L^2(\Omega)) \times C([0, \infty]; X^*()) \) for every \( 1 \leq \nu < \frac{N}{N-\delta} \).

5 - Uniqueness. If we consider the system
\[
\begin{aligned}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div} (z \tau) &= -\nu \cdot \nabla \theta \quad \text{in } Q, \\
\frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Sigma, \\
y' - A_r y + P_r \left( \sum_i \delta_i (z_i y) \right) &= \frac{P_r(x \beta \tau), \quad y(0) = 0,}
\end{aligned}
\]

we can apply the above fixed point method to prove that \( \tau \equiv 0 \) is the unique solution to this system in \( L^2(0, T; L^\infty(\Omega)) \), for all \((\tilde{\tau}, \tilde{r})\) obeying \( \tilde{r} > \tilde{\sigma}, \tilde{r} > \alpha', r > m', r > \alpha' \). \( \square \)

Remark 3.1 When \( \tau_0 = 0 \) and \( y_0 = 0 \), we can prove that the mapping that associates \((\tau(T), y(T))\) with \( u \) is continuous from \( L^2(0, T; L^\infty(\Omega)) \) into \( L^2(\Omega) \times X^*(\Omega) \). (see [10]) This result will be useful to prove Theorem 4.3.

Remark 3.2 Since \((\tau, y)\) is the solution to the system
\[
\begin{aligned}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau &= u \delta_0 - y \cdot \nabla \theta + \text{div} (z \tau) \quad \text{in } Q, \\
\frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Sigma, \\
y' - A_r y + P_r \left( \sum_i \delta_i (z_i y) \right) &= \frac{P_r(x \beta \tau), \quad y(0) = 0,}
\end{aligned}
\]

we can see that \((\tau, y)\) belongs to \( L^1(0, T; W^{1,1}(\Omega)) \times L^1(0, T; W^{1,1}(\Omega)) \).

4 Optimality conditions for \((P)\)

4.1 Existence of solutions to \((P)\)

Theorem 4.1 Suppose that \( 1 < s < \frac{N}{N-\delta} \) and \( K_U \) is a closed convex subset of \( L^2(0, T) \). Then the control problem \((P)\) admits solutions.

Proof. Let \((u_n)\) be a minimizing sequence in \( K_U \). Then \((u_n)\) is bounded in \( L^2(0, T) \). We can suppose that \((u_n)\) converges to some \( u \in K_U \) for the weak-star topology of \( L^2(0, T) \). Due to Theorem 3.1 and Remark 3.1, the sequence \((\tau_n)\) is bounded in \( L^\infty(0, T; L^\infty(\Omega)) \), \((\nu_n)\) is bounded in \( L^\infty(\Omega) \), \((y_n)\) is bounded in \( C([0, T]; X^*(\Omega)) \). Therefore we can suppose that \((\tau_n)\) converges to some \( \tau \) for the weak-star topology of \( L^\infty(0, T; L^\infty(\Omega)) \), and \((y_n)\) converges to some \( y \) for the weak-star topology of \( L^\infty(0, T; X^*(\Omega)) \). We can verify that \((\tau, y)\) is the solution to the system \((S)\) corresponding to \( u \). Moreover, we can prove that \((\tau_n(T))\) converges to \( \tau(T) \) for the weak topology of \( L^\infty(\Omega) \), and \((y_n(T))\) converges to \( y(T) \) for the weak topology of \( L^\infty(\Omega) \). By classical arguments, we deduce that \((\tilde{u}, \tilde{y}, \tilde{\tau})\) is a solution of \((P)\). \( \square \)

4.2 Regularity results for the adjoint system

Let \((\sigma, w)\) be a weak solution to the adjoint system:
\[
\begin{aligned}
-\frac{\partial \sigma}{\partial t} - \kappa \Delta \sigma - z \cdot \nabla \sigma &= \beta \cdot w \quad \text{in } Q, \\
\frac{\partial \sigma}{\partial n} &= 0 \quad \text{on } \Sigma, \\
y' - A_r w &= P_r((z \cdot \nabla) w + (\nabla w)^T z) + P_r(\sigma \nabla \theta), \quad w(T) = 0.
\end{aligned}
\]

Let \((\tau_n, y_n, u)\) be a solution to \((P)\). If \( \tau(T) = \tau_n(T) - \tau_d \) for all \( 1 < s < \frac{N}{N-\delta} \) and \( w(T) = 0 \), then \((A)\) is the adjoint system associated with \((\tilde{\tau}, \tilde{y})\).

In the sequel, we make the additional assumption:
\[ s < 1 + \frac{2q}{\left( \frac{N}{N-\delta} - 2 \right) \max(m', \alpha', q')} \]
(7)

For example, in the case where \( N = 2, q = \infty, m = 4, m = 4, \alpha = \frac{1}{2}, \alpha = \frac{1}{2}, \) all the assumptions are satisfied for \( p = \infty \), and for all \( 1 < s < \infty \). In the case where \( N = 3, q = \infty, m = 8, m = 4, \alpha = \frac{15}{8}, \alpha = \frac{15}{8}, \) all the assumptions are satisfied for \( p = 3 \), and for all \( 1 < s < \frac{15}{8} \).

Theorem 4.2 Suppose that \( \sigma_T \) belongs to \( L^s(\Omega) \) and \( w_T \) belongs to \( X^{s/2}(\Omega) \) for all \( 1 < s < \frac{N}{N-\delta} \). Under assumption (1)-(3) and (7), the adjoint state \( \sigma \) belongs to \( L^s(0, T; C(\overline{\Omega})) \).

Proof. Due to assumption (7), there exists \( 1 < \varepsilon < \frac{N}{N-\delta} \) satisfying \( \frac{N}{2} < \max(m', \alpha', q') \). We can find a
pair $(\tilde{t}, r)$ satisfying:
\[
\min(2, \frac{2r}{N}) \geq \tilde{t} > \max(m', \delta', q') , \quad r \geq \epsilon , \quad r > N ,
\]
\[
1 + \frac{N}{2r} < \frac{1}{\tilde{t}} + \frac{N}{2r} + 1 ,
\]
We can choose $(\delta, d)$ such that:
\[
\frac{N}{2r} \leq \frac{N}{2d} \leq \frac{N}{2r} + \frac{1}{2} ,
\]
\[
\frac{1}{d} + \frac{N}{2r} < \frac{1}{d} + \frac{N}{2\alpha} + \frac{N}{2d} < \frac{N}{2r} + 1 ,
\]
Then we have
\[
\frac{1}{r} \leq \frac{1}{\delta} \Rightarrow \frac{1}{r} \leq \frac{1}{d} \leq \frac{1}{\alpha} \leq \frac{1}{r} < 1 - \frac{1}{\alpha} .
\]

1 - As in the proof of Theorem 3.1, we prove that the system admits a solution $\sigma$ in $L^2(T-T_0, T; W^{1, r}((\Omega)))$ for $\tilde{t}$ small enough. Let $\xi \in L^2(T-T_0, T; W^{1, r}((\Omega)))$, and $(\sigma, w, w')$ be the solution to the system:
\[
\frac{\partial \sigma}{\partial t} - \kappa \Delta \sigma - z \cdot \nabla \sigma = \beta \cdot w \quad \text{in} \quad \Omega \times (T-T_0, T],
\]
\[
\frac{\partial \sigma}{\partial n} = 0 \quad \text{on} \quad \Sigma ,
\]
\[
-u' - \sigma_r - P_r((z \cdot \nabla)w + (\nabla w)Tz) = -\sigma_r(\xi \Delta \nabla \sigma) ,
\]
\[
w(T) = P_w(T).
\]

Since $\frac{1}{r} \geq 0 > \frac{1}{\delta} - \frac{1}{N}$, we have $\xi \in L^2(T-T_0, T; W^{1, r}((\Omega))) \hookrightarrow L^2(T-T_0, T; L^2((\Omega)))$. Moreover $\frac{1}{r} + \frac{1}{\alpha} < 1$ and $\frac{1}{\alpha} + \frac{1}{\delta} < 1$, then $\xi \Delta \nabla \sigma$ belongs to $L^2(T-T_0, T; L^2((\Omega)))$. From Proposition 2.3, we can deduce that $w$ belongs to $L^6(T-T_0, T; W^{1, r}((\Omega)))$.

Let $\xi_1$ and $\xi_2$ belong to $L^2(T-T_0, T; W^{1, r}((\Omega)))$. Still with Proposition 2.3, we can write
\[
\| \sigma_1 - \sigma_2 \|_{L^2(T-T_0, T; W^{1, r}((\Omega)))} \leq C \| \xi_1 - \xi_2 \|_{L^2(T-T_0, T; W^{1, r}((\Omega)))},
\]
where $C_2$ can be chosen depending on $T$, but independent of $\tilde{t}$. The mapping $t \mapsto C_2(\int_0^\infty \int_{\Omega} \| \theta(x, t) \|^{\frac{p}{2}}(\Omega) d\tau) \frac{d\tau}{\tau}$ is absolutely continuous, then there exists $\tilde{t} > 0$ such that $C_2(\int_0^{\frac{1}{\tilde{t}r}} \| \theta(x, t) \|^{\frac{p}{2}}(\Omega) d\tau) \frac{d\tau}{\tau} \leq C = \frac{1}{4}$ for all $t \in [0, T]$. Thus the mapping $\tilde{t} \mapsto \sigma_\tilde{t}$ is a contraction in the Banach space $L^2(T-T_0, T; W^{1, r}((\Omega)))$.

2 - We can proceed as in the proof of Theorem 3.1 to prove that the global $(\sigma, w)$ exists and is unique in $L^2(0, T; W^{1, r}((\Omega))) \times L^2(0, T; \{W^{1, r}((\Omega))\}^N \times \{X_r((\Omega))\})$.

3 - Since $\tilde{t} \geq \tilde{t}'$ and $r > N$, then $\sigma$ belongs to $L^2(0, T; W^{1, r}((\Omega))) \hookrightarrow L^2(0, T; C((\Omega)))$.

4.3 Optimality conditions for $(P)$

**Proposition 4.1** Let $(\sigma, w)$ be the solution to the adjoint system $(\mathcal{A})$, where $\sigma(T)$ belongs to $L^2(\Omega)$, $w(T)$ belongs to $L^2((\Omega))^N$ for all $1 < \epsilon < \frac{N}{2}$. Let $(\tau, y)$ be the solution to the system
\[
\begin{cases}
\frac{\partial \tau}{\partial t} - \kappa \Delta \tau + \text{div}(z \tau) = u_\delta - y \cdot \nabla \theta \quad \text{in} \quad Q ,
\quad \tau(0) = 0 \quad \text{on} \quad \Sigma ,
\quad y' - A_r y + P_r((z \cdot \nabla)w + (\nabla w)Tz) + P_r(\xi(\Delta \nabla \sigma)) = P_r(\beta \tau) ,
\end{cases}
\]
\[
\text{and} \quad y(0) = 0 .
\]

Then we have the Green formula
\[
\int_0^T u \tau(b(t)) dt = \int_0^T \sigma_\tau(x)(y(x, T)) dx + \int_\Omega w_T(x) (y(x, T)) dx.
\]

**Proof.** Since $1 < \epsilon < \frac{N}{2}$, $y(T)$ belongs to $(L^2(\Omega))^N$, and $w_T$ belongs to $(L^2((\Omega))^N$. Thus $w_T \cdot y(T)$ belongs to $L^2(\Omega)$. We also have $\sigma_T(T) \in L(\Omega)$, with $(\sigma_T^k, k)$ be a sequence of regular functions converging to $\sigma_T$ in $L^p((\Omega))^N$. Let $d_k$ be the solution to the system:
\[
\frac{\partial \sigma}{\partial t} - \kappa \Delta \sigma - z \cdot \nabla \sigma = \beta \cdot w \quad \text{in} \quad Q ,
\]
\[
\frac{\partial \sigma}{\partial n} = 0 \quad \text{on} \quad \Sigma ,
\]
\[
-w' - A_r y + P_r((z \cdot \nabla)w + (\nabla w)Tz) + P_r(\xi(\Delta \nabla \sigma)) = P_r(\beta \tau) ,
\]
\[
\text{and} \quad y(0) = 0 .
\]

Using the Green formula in [10], Proposition 3.3,
\[
\int_\Omega \sigma_T^k(\tau(x), T) dx + \int_0^T \tau(\beta \cdot w)(x) dx dt = \int_0^T u \sigma_T^k b(t) dt - \int_0^T y(\nabla \theta \sigma_T)(x) dx dt ,
\]
and
\[
\int_\Omega \sigma_T^k(\tau(x), T) dx + \int_0^T \tau(\beta \cdot w)(x) dx dt = \int_0^T u \sigma_T^k b(t) dt .
\]

Thus we obtain the Green formula (8) by passing to the limit in (9). □

Necessary optimality conditions are stated in the following theorem.

**Theorem 4.3** If $u$ is an optimal control for problem $(P)$, and $(\tau_u, y_u)$ is the corresponding solution of the system $(\mathcal{S})$, then
\[
\int_0^T (v - u)(\sigma(b(t)) \geq 0 ,
\]
\[
\text{where} \quad (v - u) \quad \text{is the optimal control}.
\]
for all $v \in K_U$, where $\sigma$ is the solution of the adjoint system

\[ (A) \text{ when } \sigma_T = s|\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d), \text{ and } w_T = s|y_u(T) - y_d|^{r-2}(y_u(T) - y_d). \]

**Proof.** Let $v$ be in $K_U$, $\lambda > 0$, and denote by $(\tau_v, y_v)$ the solution of $(S)$ corresponding to $u + \lambda(v - u)$.

1. We set $\tilde{\pi} = \tau_v - \tau_d$, and $\pi = y_v - y_d$, then $(\tilde{\pi}, \pi)$ is the solution to the system:

\[
\begin{aligned}
\frac{\partial \tilde{\pi}}{\partial t} - \kappa \Delta \tilde{\pi} + \text{div } (z \tilde{\pi}) &= \lambda(v - u) \delta_0 - \pi \cdot \nabla \theta \text{ in } Q, \\
\frac{\partial \pi}{\partial n} &= 0 \text{ on } \Sigma, \quad \tilde{\pi}(0) = 0 \text{ in } \Omega, \\
\pi' &= A_v \pi + P_r(\sum_i \delta_i(z_{i,\pi})) + P_r(\sum_i \delta_i(z_{i,\pi})) \text{ in } L^{1,\infty}(\Omega). \\
&= P_r(\beta \tilde{\pi}), \quad \pi(0) = 0.
\end{aligned}
\]

Since $s < \frac{N}{N - r - 2}$, applying Theorem 3.1, we obtain $(\tilde{\pi}, \pi) \in C([0, T]; L^1(\Omega)) \times C([0, T]; X^r(\Omega))$. From Remark 3.1, we can write the estimate:

\[
\begin{aligned}
\|\tau_v(T) - \tau_d(T)\|_{L^1(\Omega)} + \|y_v(T) - y_d(T)\|_{L^1(\Omega)} &
\leq C \lambda \|v - u\|_{L^2(u, T)}.
\end{aligned}
\]

2. We want to calculate the gradient of the functional $I$. If we set $G(u) = I(\tau_v, y_v)$, from the convexity of the mapping $y \mapsto \int_\Omega |\pi(T)| - y_a|_{dx}$, it follows that:

\[
\begin{aligned}
&\lambda \int_\Omega |\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d) \tilde{\pi}(T) \, dx \\
&+ s \int_\Omega |y_u(T) - y_d|^{r-2}(y_u(T) - y_d) \cdot \tilde{\pi}(T) \, dx \\
&\leq \frac{1}{\lambda} G(u + \lambda(v - u)) - G(u) \\
&\leq s \int_\Omega |\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d) \tilde{\pi}(T) \, dx \\
&+ s \int_\Omega |y_u(T) - y_d|^{r-2}(y_u(T) - y_d) \cdot \tilde{\pi}(T) \, dx.
\end{aligned}
\]

We set $\sigma_T = s|\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d)$, and $\sigma_T^\lambda = s|\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d)$. Due to Step 1, it follows that $\sigma_T$ and $\sigma_T^\lambda$ belong to $L^{1,\infty}(\Omega)$, and that $\int_\Omega \sigma_T \tilde{\pi}(T) \, dx \rightarrow \int_\Omega \sigma_T^\lambda \tilde{\pi}(T) \, dx$ as $\lambda$ tends to zero.

Set $w_T = s|y_u(T) - y_d|^{r-2}(y_u(T) - y_d)$, and $w_T^\lambda = s|y_u(T) - y_d|^{r-2}(y_u(T) - y_d)$. We also have $w_T$ and $w_T^\lambda$ belong to $(L^{1,\infty}(\Omega))^N$, and that $\int_\Omega \lambda \omega \cdot \pi(T) \, dx \rightarrow \int_\Omega \lambda \omega \cdot \pi(T) \, dx$ as $\lambda$ tends to zero.

Thanks to the above calculations, we obtain:

\[
G'(u)(v - u) = s \int_\Omega |\tau_u(T) - \tau_d|^{r-2}(\tau_u(T) - \tau_d) \tilde{\pi}(T) \, dx \\
+ s \int_\Omega |y_u(T) - y_d|^{r-2}(y_u(T) - y_d) \cdot \pi(T) \, dx
\]

Finally, we can use the Green formula (8) to complete the proof. \[\square\]