Distributed parameter systems spreadability and evolving interfaces

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Abstract

The notion of spreadability was introduced by A. El Jaï and al. 1994 [8, 9] to describe some biogeographical problems. An extension (A-spreadability) was given in 1998, Bernoussi [5]. The aim of this paper is to explore some interesting connections of this notion with wavefronts [11] and level set [13]. Also the spray control problem is investigated in connection with tracking interfaces. The results are illustrated by a variety of examples.

1 Introduction

A wide literature is devoted to the problems of analysis and control of DPS and many concepts have been studied and explored. Recently, the concept of spreadability has been introduced by El Jaï [8, 9] to describe expansion phenomena. The basic definition assumes that a property $\mathcal{P}$ is given; then a system $(S)$ is said to be spreadable in (space) during a time interval $I$ if the family of sets $(\omega_t)_{t \in I}$, where the property is satisfied at time $t$, is increasing in the inclusion sense. A weaker concept has been introduced in 97 [A. El Jaï and D. Ucinski, [15]], which requires increasing subdomains where the state of the system only needs to satisfy the particular property approximatively. In some applications it is more natural to consider that the property is expanding in space but without the inclusion, i.e. the measure of the domains $(\omega_t)_{t \in I}$ is increasing [Bernoussi, [5]].

In some cases, the sets can be described by their boundaries. The aim of this paper is to explore some interesting connections of spreadability with wavefronts [Murray, [11]] and level sets [Sethian, [13]]. We will assume that the basic system is modeled by partial differential equations. However it may well be that a more practical approach is to model such systems by cellular automata [El Yacoubi, [10]]. It has been shown that such models may be helpful for implementation of spreadability concept. Also the spray control problems is investigated in connection with tracking interfaces. The results are illustrated by a variety of examples.

2 Spreadability

Let $(S)$ be a dynamical system defined in $\Omega \subset \mathbb{R}^n$ with a state denoted by $z(x,t) = z(x,t,t_0,z_0)$, $t \geq t_0$, $x \in \Omega$, $t_0$ is the initial time and $z_0(x) = z(x,t_0)$. We assume the following evolution property

$$z(x,t,t_0,z_0) = z(x,t,s,z(s,s,t_0,z_0)), \quad t_0 \leq s \leq t.$$  \hspace{1cm} (2-1)

Let $\omega_t$ be the set of all points $x \in \Omega$ such that the state $z(x,t)$ satisfies a given property $\mathcal{P}$ at time $t$.

$$\omega_t = \omega_t(t_0,z_0;\mathcal{P}) = \{x \in \Omega \mid \mathcal{P}z(x,t,t_0,z_0)\} \hspace{1cm} (2-2)$$

We are interested in the study of the evolution in time of the sets $\omega_t$.

2-a Basic approach of spreadability

Firstly we recall the definition of the spreadability as defined in [9]. Given a property $\mathcal{P}$, we assume that the subdomains $\omega_t$ are defined in (2-2).

Definition 2.1

- The system $(S)$ is said to be $\mathcal{P}$-spreadable (respectively strictly spreadable) from $\omega_{t_0}$ during the time horizon $I = [t_0,T]$ ($T > t_0$) if the family of sets $(\omega_t)_{t \in I}$ is increasing (respectively strictly increasing) in the sense that $\omega_s \subset \omega_t$, $0 \leq s \leq t \leq T$.

- The system $(S)$ is said to be $\mathcal{P}$-resorbable if the family of sets $(\omega_t)_{t \in I}$ is decreasing.

- In the particular case where $\mathcal{P}$ is the property that $z(x,t) = 0$ the system $(S)$ is said to be null-spreadable.
In [8] various examples are presented and a characterization of spreadable systems via a nonlinear equation is given. We give an extension of the above definition as stated by Bernoussi, 1998 [3, 5].

2-b New approach of spreadability

Instead of requiring that the sets \( \omega_t \) increase as in definition 2.1 here we only assume that the measure of the sets \( (\omega_t)_{t \in I} \) increases. The measure is the usual Lebesgue one.

\[
\mu(\omega_t) = \int_{\omega_t} dx.
\]

Clearly the results will depend strongly on \( \omega_{t_0} \) and it is possible that spreadability may occur even when \( \mu(\omega_{t_0}) = 0 \), for example when \( \omega_{t_0} \) is a single point \( b, b \in \Omega \) [5].

Definition 2.2 The system \((S)\) is said to be \(A\)-spreadable (or spreadable in measure) from \( \omega_{t_0} \) during the time interval \( I = [t_0, T], \) \( T > t_0, \) if:

\[
\mu(\omega_t \setminus \omega_s) \geq \mu(\omega_s \setminus \omega_t) \quad \forall t, s, \quad t_0 \leq s \leq t \leq T.
\]

Remark 2.3 The above condition means that as time increases what is won is bigger than that which is lost and we have written it in this way rather than saying that the measure increases since it may be that the measure is infinite.

It is clear that if a system is spreadable (in the usual sense), then it is \(A\)-spreadable. The converse is not true [5].

In the following we only consider the case of null-\(A\)-spreadability; the case where \( P \) is the property that \( z(x,t) = 0 \). For a characterization of such systems it is necessary to introduce a family of functions which describe in some sense the evolution of the subdomains \( (\omega_t) \). We give the main features of this characterization. For more details, see [3, 5].

Definition 2.4 A family of mappings \((F(.,t,s))_{t \geq s}\), where \( F: \Omega \times \Delta \longrightarrow \Omega \), is said to be

adapted to the evolution of the system \((S)\) if there exists a function \( \eta: \Omega \times \Delta \longrightarrow \mathbb{R} \) such that

\[
z[F(x,t,s),t] = \eta(x,t,s)z(x,s), \quad t \geq s \geq t_0, \quad x \in \Omega.
\]

(2-3)

where \( \Delta = \{(t,s) \in I \times I, t \geq s\}. \)

We have the following result

Proposition 2.5 The family \((F(.,t,s))_{t \geq s}\) is adapted to the evolution of the system \((S)\) if and only if there exists a function \( \xi(x,s) \) such that

\[
\xi(x,s)z(x,s) = \int_s^T z^2[F(x,t,s),t] \, dt, \quad s \geq t_0, \quad x \in \Omega.
\]

(2-4)

Proof. If \( \xi(x,s) \) satisfies (2-4), then \( z(x,s) = 0 \) \( \Rightarrow \)

\[
\int_s^T z^2[F(x,t,s),t] \, dt = 0 \quad \text{and hence } z[F(x,t,s),t] = 0, \quad \forall t \geq s.
\]

So we consider \( \eta(x,t,s) = \text{constant} \) for \( x \in \omega_s \) and \( \eta(x,t,s) = z[F(x,t,s),t]/z(x,s) \) otherwise.

Conversely if \((F(.,t,s))_{t \geq s}\) is adapted to the system \((S)\), consider

\[
\xi(x,s) = \begin{cases} \int_s^T \frac{z^2[F(x,t,s),t]}{z(x,s)} \, dt & \text{if } x \notin \omega_s \\ \text{constant} & \text{if } x \in \omega_s \end{cases}
\]

Consequently we have a characterization of \(A\)-spreadable systems.

Corollary 2.6 The system \((S)\) is \(A\)-spreadable if and only if there exists a family of maps \((F(.,t,s))_{t \geq s}\) which is adapted to its evolution such that \( \mu(\omega_s, t, s) \geq \mu(\omega_t, t, s) \) for all \( t \geq s \geq t_0. \)

If \((F(.,t,s))_{t \geq s}\) is a family of diffeomorphisms of \( \Omega, \) then the system \((S)\) is \(A\)-spreadable if and only if

\[
\int_{\omega_s} [\|JF(x,t,s)\| - 1] \, dx \geq 0; \quad \forall t, s, \quad t \geq s \geq t_0
\]

(2-5)

or equivalently if \( F(\omega_s, t, s) = \omega_t \) \( \forall t, s \geq s \geq t_0. \)

where \( JF(x,t,s) \) is the Jacobian of the mapping \( F(x,t,s) \) in \( x. \)

Proof. Let \( F(x,t,s) = y \) if \( F(\omega_s, t, s) = \omega_t, \) then

\[
\mu(\omega_t) = \int_{\omega_t} dy = \int_{\omega_s} |JF(x,t,s)| \, dx
\]

by a variable change, we obtain

\[
\int_{\omega_t} dx \geq \int_{\omega_s} dx \quad \Leftrightarrow \quad \int_{\omega_t} [\|JF(x,t,s)\| - 1] \, dx \geq 0
\]

and similarly as \( \mu(\omega_s) \) is increasing, this is equivalent to

\[
\frac{\partial}{\partial t} \int_{\omega_t} [\|JF(x,t,s)\|] \, dx \geq 0
\]

If \( F(\omega_s, t, s) \subset \omega_t \) and if (2-5) is satisfied, then \((S)\) is \(A\)-spreadable because

\[
\int_{\omega_t} dx \geq \int_{F(\omega_s, t, s)} dx = \int_{\omega_s} |JF(x,t,s)| \, dx
\]
Remark 2.7 In the case where $F(., t, s) = Id_{\Omega}$ (injection of $\omega_s$ in $\omega_t$) then we retrieve the usual spreadability [8]. In this case $(S)$ is $A$-spreadable if and only if $(Id_{\Omega})$ is adapted to $(S)$, i.e. (2-4) has a solution $\xi(x, t, s)$ which is $R$-valued (if the solution is $R^*$-valued, then $(S)$ is stationary in the sense of $\omega_t = \omega_s$).

Notice that in this case, the conditions (2-5) and (2-6) are satisfied because

$$\int_{\omega_s} |JF(., t, s)| = 1 \text{ in } \Omega$$

and considering $(F(., t, s))_{t \geq s}$ such that $\int_{\omega_t} |JF(x, t, s)| - 1 \ dx \geq 0$ we obtain $A$-spreadability.

The choice of the family $(F(., t, s))_{t \geq s}$ is then very important.

As previously mentioned, the family of mappings which is adapted to the evolution of a given system plays an important role regarding spreadability because it allows to track the geometry evolution of the subdomains $(\omega_t)$. We have given a simple characterization of such family but their explicit determination remains difficult. However in some particular situations one may have more information on the evolution, i.e. the speed propagation, for example and then can find the geometric evolution of such subdomains $\omega_t$. This also occurs particularly if we know more on the interface evolution.

The case where the speed propagation is constant is interesting for certain biogeographic phenomena which are characterized by their periodicity [11] and which propagate with a constant speed. This has naturally led to the notion of wavefront as it was given in [11]. Notice that the notion of wavefront has been initially introduced to describe propagating phenomena in optics, electromagnetism, chemistry.

Thus we are naturally led to consider the spreadability and its possible connections with wavefronts and evolving interfaces.

3 Spreadability and wavefronts

The spreadability concept is defined via subdomains $(\omega_t)$ where a given property holds. Now we concentrate on the evolution of such subdomains by means of the evolution of their boundaries $(\partial \omega_t)$. We recall that a travelling wave is a wave which propagates with a nonvarying shape and a constant speed.

3-a Wavefronts as a special case of spreadability

If we consider the particular case of a Fisher equation as presented in [11] and which governs various phenomena

$$\frac{\partial z}{\partial t} = k z(1 - z) + D \frac{\partial^2 z}{\partial x^2}$$

where $k$ and $D$ are positive and constant. With the variable change $t^* = kt$ and $x^* = x(k/D)^{1/2}$, (3-1) becomes

$$\frac{\partial z}{\partial t} = z(1 - z) + \frac{\partial^2 z}{\partial x^2}$$

If we consider that the state of this equation is moving like a travelling wavefront, then there exists a constant $c$ and a function $Z$ such that

$$z(x, t) = z(x - ct) = Z(y), \quad y = x - ct.$$  (3-3)

The variable $y$ is commonly called wave variable. One important problem studied by Murray in [11] concerns the existence of travelling waves which characterize the evolution of physical phenomena. If we consider the equation (3-2), then a travelling wave is any solution in the form (3-3). The derivatives of (3-3) are

$$\ddot{Z} + c\dot{Z} + Z(1 - Z) = 0$$

where $\dot{Z} = \frac{dZ}{dy}$ and $\ddot{Z} = \frac{d^2Z}{dy^2}$. Thus the determination of solutions $z(x, t)$ which evolve as wavefronts necessitates solving equation (3-5). This will lead to the possible values of $c$ for which there exist solutions $Z$ of (3-5). For the problems for which this notion has been introduced, we only focus on particular solutions which are interesting from physical point of view. Thus we get the associated particular values of the speed $c$.

Let us explore how this notion is connected to $A$-spreadability. Consider a family of mappings $(F(., s, t))_{s \geq t}$ adapted to the evolution of the system $(S)$ given by (3-2). With the properties of $(F(., s, t))_{s \geq t}$, the equation (2-3) is equivalent to

$$z[F(x, t, t_0), t] = \eta(x, t, t_0) z(x, t_0)$$

Without loss of generality, take $t_0 = 0$, then (3-6) becomes

$$z[F(x, t), t] = \eta(x, t) z(x, 0)$$

where $F(x, t) = F(x, t, 0)$ and $\eta(x, t) = \eta(x, t, 0)$.

Assuming that the system $(S)$ evolves by generating a travelling wavefront which propagates with a constant speed $c$, the relation (3-3) shows that $F(x, t) = x + ct$ is adapted to the evolution of $(S)$ (solution of equation (3-7) with $\eta(x, t) = 1$).

In Proposition 2.5 we gave a characterization of the family of mappings adapted to the system $(S)$ based on a differential equation (this results from the particular situation
where one parameter is considered for the spreadability, i.e. the constant \( c \).

Thus depending on the nature of the considered property \( \mathcal{P} \), and the shape of the subdomains \( (\omega_t)_{t \in \mathbb{R}} \), we are able to formulate travelling wavefront problems as particular situations of \( \mathcal{A} \)-spreadability. This can be illustrated, for example, by the case of the superposition of two wavefronts which are travelling in two opposite directions (axisymmetric form of Fisher equation), see [11]. Other examples are studied in [11] such as the approximate chronological spread of the Black Death in Europe from 1347-50 or the case of circular waves generated by pacemaker nuclei in the Belousov-Zhabotinski reaction [11]. This shows that the wavefront notion is a very particular situation of \( \mathcal{A} \)-spreadability.

Obviously there exist systems which evolve in a general way and do not correspond to any wavefront or combination of wavefronts and which are \( \mathcal{A} \)-spreadable. Let us illustrate such systems with the following example.

**Example 3.1** Consider the set \( \Omega = [0, a] \) with \( a > 0 \) sufficiently large and the dynamical system (8) described by the differential equation

\[
\begin{align*}
\frac{\partial z}{\partial t}(x, t) &= b_1(x, t)z(x, t) \quad t > 0, \ 0 < x < a \\
z(x, 0) &= z_0(x) \quad 0 < x < a
\end{align*}
\]

where

\[
z_0(x) = \begin{cases} 
0 & \text{if } \ 0 < x \leq 1 \\
\exp\left(\frac{1}{1 - x}\right) & \text{if } \ x > 1
\end{cases}
\]

and

\[
b_1(x, t) = \begin{cases} 
0 & \text{if } \ 1 \leq \frac{x + 1}{t + 1} \leq 2 \\
\frac{x + 1}{2(t - x + 1)} & \text{if } \ \frac{x + 1}{t + 1} > 2
\end{cases}
\]

For \( x \in \Omega \) and \( t > 0 \), the solution is given by

\[
z(x, t) = \begin{cases} 
0 & \text{if } \ 1 \leq \frac{x + 1}{t + 1} \leq 2 \\
\exp\left(\frac{1}{2} \frac{x + 1}{t + 1}\right) & \text{if } \ 2 < \frac{x + 1}{t + 1}
\end{cases}
\]

(3-9)

Consider the null \( \mathcal{A} \)-spreadability case with

\[
\omega_t = \begin{cases} 
\{ x \in \Omega \mid z(x, t) = 0 \} & \\
\{ x \in \Omega \mid 1 \leq \frac{x + 1}{t + 1} \leq 2 \}
\end{cases} \text{ for } t \leq \frac{a - 1}{2}
\]

If the system starts from \( \omega_0 = [0, 1] \subset \Omega \) (a sufficiently large), we have \( \mu(\omega_0) = 1 \) and \( \mu(\omega_t) = t + 1 \) is increasing, thus the system is null-\( \mathcal{A} \)-spreadable from \( \omega_0 \). Denote \( u(x, t) = \frac{x + 1}{t + 1} \), then we have

\[
z(x, t) = z_0[u(x, t) - 1] = z_0\left(\frac{x - t}{t + 1}\right), \quad u(x, t) \geq 1.
\]

Let \( y = \frac{x - t}{t + 1} \), then the solution satisfies the equation

\[
\begin{cases}
z(x, t) = z_0(y) \\
y(x, t) = \frac{x - t}{t + 1}
\end{cases}
\]

(3-10)

Thus the system is null \( \mathcal{A} \)-spreadable and the solution satisfies

(3-10) (displacement of the spreadability front) but it does not evolve like a travelling wavefront as defined above. This shows that the \( \mathcal{A} \)-spreadability concept allows the representation of more complicated phenomena in the spatial evolution which do not necessarily correspond to a travelling wavefront. Murray in [11] notices that, even for very particular problems, it is generally difficult to characterize the wavefront solutions.

**Remark 3.2** In \( \mathcal{A} \)-spreadability, the initial condition is very important as illustrated in [5]. It also plays an important role for wavefront problems, even though it has not been explicitly clarified. For the Fisher equation (3-2) for example, the equation (3-5) gives the existence necessary condition for travelling wavefront solutions. So there are very particular classes of initial conditions which lead to travelling wavefront solutions, [11].

The notion of travelling wave is a particular situation of spreadability and is based on a constant propagation speed and a non-varying shape. For the case of spreadability, the geometry of the expanding subdomains \( (\omega_t) \) may vary with a completely general shape. For the speed propagation of these subdomains we are then naturally led to define the spread velocity.

### 3-b Spread velocity and wavefront

Various definitions of this notion may be considered but one must be careful because spreadability may generate a change in the topology of the subdomains, sharp corners may appear as the systems evolves. Thus a definition which is based on the regularity of the boundaries \( (\partial \omega_t) \) may lead to difficulties.

If we consider the Fisher equation (3-1) in the one-dimensional case, we have seen that it evolves as a travelling wavefront with a constant speed \( c \) in the positive \( x \)-axis direction. In this case we have

\[
\mu(\omega_t) = \mu(\omega_{t_0}) = K \quad (\text{constant}) \quad \forall t,
\]
which means that the measure of $\omega_t$ is constant. Consider now the two-dimensional case with circular waves, the front of which evolves with a constant speed $c$ from the center 0 and $\omega_t$ are disks of radius $ct$ centered at the origin 0; then

$$
\mu(\omega_t) = \pi c^2 t^2
$$

The measure of $\omega_t$ is increasing with time. It is then interesting to consider the spread velocity, which will be called "growth speed" and to compare it with the wavefront speed.

Spreadability as defined in [8, 9] and [5] may be considered as an evolution phenomenon in the global sense. Thus we can consider the spread velocity at time $t$ as defined by

$$
V(t) = \frac{d\mu(\omega_t)}{dt}
$$

(3-11)

**Remark 3.3:** From the above definition, it is clear that

1. The system $(S)$ is $A$-spreadable if and only if

$$
V(t) = \frac{d\mu(\omega_t)}{dt} \geq 0
$$

2. If $(F(., t, s))_{t \geq s}$ is a family of diffeomorphisms which is adapted to $(S)$ and $F(\omega_s, t, s) = \omega_t$ $\forall t \geq s$, then

$$
V(t) = \frac{\partial}{\partial t} \int_{\omega_0} |\mathcal{J}F(x, t, t_0)| dx
$$

Thus as the $A$-spreadability holds when $\frac{d\mu(\omega_t)}{dt} \geq 0$, we obtain the characterization given in [5],

$$
\frac{\partial}{\partial t} \int_{\omega_0} |\mathcal{J}F(x, t, t_0)| dx \geq 0
$$

3. The spread velocity as introduced above by (3-11) is different from the propagation speed of the wavefront.

With this definition we can consider spreadability as a displacement of the front of $(\omega_t)$ and a growth in measure. Let us denote by $v(x, t)$ the displacement speed of the property which defines the $(\omega_t)$, for all $x$ and $V(t)$ the growth (in measure) speed. If $(F(., t, s))_{t \geq s}$ is a family of diffeomorphisms which is adapted to the system $(S)$ such that $F(\omega_s, t, s) = \omega_t$, then the growth speed is

$$
v(x, t) = \frac{\partial}{\partial t} F(x, t, t_0)
$$

We have noticed that if a system $(S)$ admits an adapted family of mappings, then this family is not unique [5]. However, in the one-dimensional case, we can choose particular adapted families of mappings. We have the result.

**Proposition 3.4:** Consider a system $(S)$ defined on $\Omega$ in the case where the space dimension $n = 1$, and assume that the $\omega_t$ are connected sets for all $t \in I$, then there exist two functions $a(t)$ and $b(t)$ such that the family of mappings $F(x, t) = a(t)x + b(t)$ is adapted to the system $(S)$.

**Proof.** Without loss of generality we consider $\omega_0 = [0, 1]$ and $\omega_t = [\alpha(t), \beta(t)]$, then $b(t) = \alpha(t)$ and $a(t) = \beta(t) - \alpha(t)$ and $\mu(\omega_t) = a(t)$. The displacement speed for any $x$ is

$$
v(x, t) = \frac{\partial}{\partial t} F(x, t) = a'(t)x + b'(t)
$$

The displacement speed for 0 and 1 is then

$$
\begin{cases}
v(0, t) = b'(t) \\
v(1, t) = a'(t) + b'(t)
\end{cases}
$$

The initial conditions are $F(0, 0) = 0 = b(0)$ and $F(1, 0) = a(0) = 1$, thus

$$
\begin{cases}
z[F(0, t), t] = 0 \\
z[F(1, t), t] = 0
\end{cases}
$$

which gives

$$
\begin{cases}
z(b(t), t) = 0 \forall t > 0 \\
z(a(t) + b(t), t) = 0 \forall t > 0
\end{cases}
$$

The system $(S)$ is $A$-spreadable if $a'(t) \geq 0$, because $\mu(\omega_t) = a(t)$.

If we consider Example 3.1, then it is easy to see that the family [5]

$$
F(x, t, s) = \begin{cases}
t + 1 \\s + 1
\end{cases} x + t + 1 \\s + 1
$$

if $x \in [s, 2s + 1]$

$$
\begin{cases}
2t + 1 \\2s + 1
\end{cases} x
$$

if $x > 2s + 1$

is adapted to $(S)$ and we have $F(x, t, 0) = (t + 1)x + t$, $x \in [0, 1]$, $F(x, t, 0) = (2t + 1)x$, $x > 1$. The displacement speed is then

$$
v(x, t) = \frac{\partial F}{\partial t}(x, t, 0) = x + 1, \ x \in [0, 1].
$$

Note that, in this case, the points 0 and 1 move with the speed 1 and 2 respectively. On an other hand

$$
\frac{\partial}{\partial t} \int_{\omega_0} |\mathcal{J}F(x, t, 0)| dx_0 = \frac{\partial}{\partial t} \int_0^1 (t + 1) dx = 1 \geq 0
$$

which is the growth speed of the $\omega_t$ measure. We have seen that, in this case, $\omega_t = [t, 2t+1]$ and then $\mu(\omega_t) = t+1$. 

In summary the $A$-spreadability of a system is characterized by the adapted family of mappings $(F(\cdot, t, s))_{t \geq s}$. These mappings allow one to track the evolving subdomains $(\omega_t)_{t \in I}$ but they are not unique. Finally in the one-dimensional case, one can exhibit specific classes of adapted mappings. In the case of a space dimension $\geq 2$, we can track the spreadability by means of tracking the boundaries which may be considered as evolving interfaces.

4 Spreadability and level set methods

4-a Level set principle

We recall that an interface is a region which divides a domain into two subdomains in which the states of the system are different. We are interested in tracking a moving interface which depends on the system dynamics. For interface problems we generally assume that the interface speed is known. The knowledge of this speed in the case of spreadable dynamical systems is a quite complicated problem. Sethian 1996 [13] has suggested that the interface speed may depend on three terms $L, G$ and $I$ which are related to the local properties (local geometry, curvature), the global properties of the front (differential description of the front, system dynamics, ...) and to some independent properties which do not result from the interface itself. This interface speed is denoted by $v(L, G, I)$. Sethian has produced a model for the propagating front (evolving interface) in the case where the interface speed $v$ only depends on the local curvature and the author uses level set techniques for the developments. The author states the model formulation both in the general case and in the case where the speed remains positive (or negative); this case leads to a stationary formulation. In both cases the model is augmented by an entropy condition derived from a persistency hypothesis. Moreover the author shows that a front propagation remains smooth when the speed depends on its curvature.

Consider an hypersurface $\Gamma(t)$ of dimension $(n-1)$ which evolves in the direction of the outward normal with a speed $v$ only depending on the local curvature, $v = v(L)$, from its initial position $\Gamma(0)$. The principle of the level set method consists of embedding this propagating interface as the zero level of a higher dimensional function $\phi$. The function $\phi = \phi(x, t)$ depends on the space variable $x \in \mathbb{R}^n$ and on a second variable $t$; the function $\phi(x, t)$ is reduced to the front $\Gamma(t)$ when $\phi(x, t) = 0$. The function $\phi$ is defined by considering

$$\phi(x, 0) = \pm d$$

where $d$ is the distance from $x$ to $\Gamma(0)$. The sign $\pm$ is chosen depending on the location of the point $x$ in or outside the initial hypersurface $\Gamma(0)$. Thus the goal is to find an equation describing the evolving function $\phi(x, t)$ which contains the embedded motion of $\Gamma(t)$ as the level set $\phi = 0$. For that purpose we firstly have an initial condition $\phi(x, 0)$ given by $\Gamma(0) = \{x/\phi(x, 0) = 0\}$. Secondly we notice that the zero level set of the evolving function $\phi$ always matches the propagating hypersurface which may be written $\phi(x(t), t) = 0$, then by differentiation and in the case where $x'(t), \vec{n} = v$, with $\vec{n} = \nabla \phi/|\nabla \phi|$, we obtain [12, 13]

$$\begin{cases}
\phi_t + v|\nabla \phi| = 0 \\
\phi(x, 0) = 0
\end{cases} \text{ given}$$

(4-1)

which reduces, for certain forms of $v$, to the standard Hamilton-Jacobi equation.

Another interesting problem studied by Sethian is the evaluation of the total variation (or total oscillation) of the front. In the two-dimensional case and if the speed function $v$ depends only on the local curvature, denoted by $\kappa$, of the curve, that is, $v = v(\kappa)$. Let $(x(s, t), y(s, t))$ be a parametrization of $\Gamma(t)$ at time $t$, where $0 \leq s \leq S$ with periodic boundary conditions $(x(0, t), y(0, t)) = (x(S, t), y(S, t))$, then the total variation of the front is given by

$$Var(t) = \int_0^S \kappa(s, t) \left( \frac{y_s^2 + y_{ss}^2}{x_s^2 + y_s^2} \right)^{1/2} \, ds$$

where $\kappa = \frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}}$ is the local curvature of the curve, $x_s = \frac{\partial}{\partial s} x(s, t)$ and $x_{ss} = \frac{\partial^2}{\partial s^2} x(s, t)$.

For additional details see [13] and the references therein.

Here we characterize the geometric evolution of the front whilst for the spreadability, we have to evaluate the measure of the enclosed surface, that is to say $\mu(\omega_t)$.

4-b Level set and $A$-spreadability

4-b.1 Evolving interfaces and $A$-spreadability

The introduction of level set techniques has permitted the study of various problems related to evolving interfaces [13]. The notion of $A$-spreadability, motivated by the study of biogeographical problems, has been introduced mainly for tracking the evolution of the domains where a given property $P$ holds. For example, in pollution problems it is interesting to follow the evolution of the interface between the polluted and the non polluted domains. But it is also
interesting to know the area (volume) of the polluted domain and its evolution. As an illustrative example, consider the bidimensional case and assume that the domains \( \omega_t = \{ x \in \Omega \subset \mathbb{R}^2 \mid \mathcal{P}(x, t) \} \) with boundary \( \Gamma(t) \). We calculate the variation of the measure of \( \omega_t \) as the system evolves. For that purpose we only consider the case where the speed \( v \) depends on the curvature \( \kappa \). We can parameterize \( \Gamma(t) \) by considering

\[
\Gamma(t) = \left( \frac{x(s, t)}{y(s, t)} \right) ; \quad 0 \leq s \leq S \tag{4.2}
\]

by expressing \( x_t \) and \( y_t \) with the curvature \( \kappa \) and the speed \( v(\kappa) \) [13], we have

\[
\begin{align*}
x_t &= v(\kappa) \frac{y_s}{(x_s^2 + y_s^2)^{1/2}} \\
y_t &= -v(\kappa) \frac{x_s}{(x_s^2 + y_s^2)^{1/2}} \tag{4.3}
\end{align*}
\]

where

\[
\kappa = \kappa(s, t) = \frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}} \tag{4.4}
\]

We have the result

**Proposition 4.1** Assume \( \Gamma(t) \) sufficiently regular and \( x(s, t) \) and \( y(s, t) \) have continuous partial derivatives, then the system \( (S) \) is \( \mathcal{A} \)-spreadable if and only if

\[
\frac{1}{2} \int_0^S \left[ v(\kappa)(x_s^2 + y_s^2)^{1/2} - \kappa v(\kappa)(\frac{xx_s + yy_s}{(x_s^2 + y_s^2)^{1/2}} - \kappa v(\kappa)(y_{ss}x_s - x_{ss}y_s)\right] ds \geq 0 \quad \forall \ t \in I \tag{4.5}
\]

For the proof, see [5].

**Remark 4.2** If we assume that \( 0 \in \omega_t \) for all \( t \in I \) (this condition may be obtained by a variable change), then we can give a parameterized form for \( \Gamma(t) \)

\[
\begin{align*}
x(s, t) &= \rho(s, t) \cos(s) \\
y(s, t) &= \rho(s, t) \sin(s) \tag{4.6}
\end{align*}
\]

where \( s \in [0, 2\pi] \) and \( \rho(s, t) = \sqrt{x(s, t)^2 + y(s, t)^2} \). Denote \( \rho(s, t) = \rho \), then we have

\[
\begin{align*}
\sqrt{x_s^2 + y_s^2} &= \sqrt{\rho^2 + \rho_s^2} \\
x_{ss} + yy_s &= \rho \rho_s \\
y_{ss} - xx_s &= -\rho^2 \tag{4.7}
\end{align*}
\]

For the proof of (4.7), we differentiate (4.6) with respect to \( s \) to obtain

\[
\begin{align*}
x_s &= \rho_s \cos(s) - \rho \sin(s) \\
y_s &= \rho_s \sin(s) + \rho \cos(s) \tag{4.8}
\end{align*}
\]

By a second derivative, we have

\[
\begin{align*}
x_{ss} &= \rho_{ss} \cos(s) - 2 \rho \rho_s \sin(s) - \rho^2 \sin(s) \tag{4.9}
\end{align*}
\]

and replacing in (4.7) we have the result.

\[
\frac{d\mu(\omega_t)}{dt} = \frac{1}{2} \int_0^S [v(\kappa)(\rho^2 + \rho_s^2)^{1/2} - \kappa v(\kappa)(\rho)^2] ds \tag{4.10}
\]

or equivalently

\[
\frac{d\mu(\omega_t)}{dt} = \frac{1}{2} \int_0^S [v(\kappa)(\sqrt{\rho^2 + \rho_s^2} + \kappa \rho^2) - \kappa v(\kappa)(\rho)^2] ds \tag{4.11}
\]

Finally the system is \( \mathcal{A} \)-spreadable if

\[
\int_0^S [v(\kappa)(\sqrt{\rho^2 + \rho_s^2} + \kappa \rho^2) - \kappa v(\kappa)(\rho)^2] ds \geq 0 \quad \forall \ 0 < t < T \tag{4.12}
\]

In the concept of \( \mathcal{A} \)-spreadability, the speed is generated by the dynamics of the system. This has not been used in the above approach which may be used for all the cases where the front evolution does not depend on the system dynamics. An important particular case is when the speed propagation of the front is constant. In the next paragraph we explore this situation.

**4-b.2 Application 1: Particular cases**

**Case where \( v(\kappa) \) is constant**

If the speed \( v(\kappa) \) is positive and constant (independent of \( \kappa \)), then (4.11) gives

\[
\frac{d\mu(\omega_t)}{dt} = \frac{1}{2} \int_0^S v(\kappa)(\sqrt{\rho^2 + \rho_s^2} + \kappa \rho^2) ds \tag{4.13}
\]

In this case the system is \( \mathcal{A} \)-spreadable if

\[
\int_0^S v(\kappa)(\sqrt{\rho^2 + \rho_s^2} + \kappa \rho^2) ds \geq 0 \quad \forall \ t \in I \tag{4.14}
\]

This condition is satisfied in the case where \( \kappa \geq 0 \).

As cited in [13], geometric properties of the front may be easily derived from the level function \( \phi \) which describes the
that for such systems it is possible to characterize the \( A \)-spreadability under certain conditions on the curvature \( \kappa \) (condition (4-14)). We can use the level set techniques for finding this condition as in (4-16).

**4-b.3 Application 2: Self-similar shapes in three dimensional case**

One important problem studied by Angenent [1] and Chopp [6] concerns the existence of a surface which evolves without changes of shape "self-similar shapes". For this problem Chopp [6, 13] used the level sets methods. We will show that for such systems it is possible to characterize the \( A \)-spreadability. We consider the three dimensional case. The author had first considered an equation in \( \phi \) which characterizes the evolution of the surface in the case where the speed depends only on the mean curvature \( \kappa^\phi \).

\[
\phi_t = \kappa^\phi |\nabla \phi| \tag{4-17}
\]

Then he introduced an other function \( \psi \) defined by

\[
\psi(x, t) = \frac{\phi(x, t)}{\sigma(t) - L(t)} \tag{4-18}
\]

where \( \sigma(t) \) acts as a stretching function and \( L(t) \) will be used to pick out the level set corresponding to the front of interest. By differentiating (4-18) with respect to \( t \), we obtain

\[
\psi_t = \frac{\sigma'(t)}{\sigma(t)} [x, \nabla \psi - 2(\psi + L(t))] + \kappa^\psi |\nabla \psi| - L'(t) \tag{4-19}
\]

where \( \kappa^\psi \) is the mean curvature of the surface defined by \( \psi \) at the point \( (x, t) \). \( \sigma(t) \) is chosen such that the level set of \( \psi \) has a constant volume for all \( t \). Put

\[
\omega_t = \{ x : \phi(x, t) \leq 0 \}
\]

and finally

\[
\frac{d\mu(\omega_t)}{dt} = 3\sigma(t)^2 \mu(\omega_t) \sigma'(t) \quad \tag{4-20}
\]

So in this case, the system (5) is \( A \)-spreadable if \( \sigma'(t) \geq 0 \) for all \( t \in I \). Thus for this class of systems (surfaces) we can give a characterization of the \( A \)-spreadability using the level sets methods.

**4-c Spreadability difficulties**

In the previous section we have assumed that the front speed \( v \) is known and depends only on the curvature of the front, \( \kappa \). For the \( A \)-spreadability problem, the front speed is generated by the internal dynamics of the system. In this section we explore the possibility of determining the front speed from the knowledge of the dynamics of the system which is assumed to be \( A \)-spreadable (or \( A \)-resorbable).

For this purpose we consider a dynamical system whose state is denoted by

\[
z(x, t) = z(x, t, t_0, z_0), \quad t \geq t_0.
\]

The system is assumed to evolve in the domain \( \Omega \subset \mathbb{R}^n \) \( (n = 1, 2 \) or \( 3) \). The initial time and the initial state are denoted by \( z_0 \) and \( t_0 \) respectively. Let

\[
\omega_t = \{ x \in \Omega \mid \mathcal{P}z(x, t, t_0, z_0) \}
\]

be the subsets of \( \Omega \) in which the property \( \mathcal{P} \) is satisfied. Among the usual properties is that which defines the null-\( A \)-spreadability defined by

\[
\mathcal{P}z(x, t, t_0) \Rightarrow z(x, t, t_0) = 0 \tag{4-21}
\]

In what follows we assume that \( \omega_t \) is a subdomain of \( \mathbb{R}^2 \) the boundary of which is a closed curve denoted by \( \Gamma(t) = \partial z, t_0 \), for all \( t \).

We assume that the system is null-\( A \)-spreadable and we try to determine the front evolution speed of the system. For this purpose, we first characterize the front \( \Gamma(t) \) using the state of the system \( z(., t) \). Secondly we give geometrical properties of the front, such as the outward normal. Finally we determine the speed of the front evolution.

**4-c.1 Normal speed characterization**

As cited above, the level function \( \phi(x, t) \) characterizes the evolution of the front \( \Gamma(t) \). If the speed of the front \( \Gamma(t) \) is known and normal to \( \Gamma(t) \) equation (4-1) gives the level function \( \phi(x, t) \). Reciprocally if the level function is known, the same equation gives the normal speed at the front. In our situation we don’t know either the speed nor the level function. Thus we give a characterization of the front and calculate the normal speed of any \( x \in \Gamma(t) \) using the state of the system \( z(., t) \). We consider a particular situation where the system is \( A \)-spreadable (or \( A \)-resorbable) and there exists
a family of mappings \( F(x, t) \) \( (= F(x, t, t_0), \ t_0 = 0) \)
of \( \Omega \) which satisfies
\[
z(F(x, t), t) = z_0(x)
\]
In the case of null-spreadability, for example, the difficulty ofcharacterizing the evolving front is related to the fact that,for all \( x \in \Gamma(t) \) we have \( z(x, t) = 0 \) and \( \nabla z(x, t) = 0 \).Indeed if \( \eta(t) \) is continuous and vanishes in \( \omega \), then thelimit gives \( z(\cdot, t) \) vanishes in \( \Gamma(t) \). Similarly the gradient,\( \nabla z(\cdot, t) \) is null on \( \Gamma(t) \).

Assume that the state \( z(x, t) \) is sufficiently regular, then bydefinition of the boundary of \( \omega \), \( \partial \omega \), we have
\[
\Gamma(t) = \{ x(t) \in \Omega : z(x(t), t) = 0 \ \text{and} \ \exists \mathcal{V}(x(t)) \}, \text{ an}
\]
neighbourhood of \( x(t) \), such that
\[
\forall y \in (\mathcal{V}(x(t)) - \omega), \ \nabla z(y, t) \neq 0
\]
Then if we assume that at any point \( x(t) \in \Gamma(t) \), theoutward normal at \( \Gamma(t) \) is
\[
\vec{n}(x(t), t) = \lim_{y \to x(t), y \notin \omega} \frac{\nabla z(y, t)}{||\nabla z(y, t)||}
\]
we have the following result

**Proposition 4.3** Assume the system null-A-spreadable (or\( \mathcal{A} \)-resorbable) and the state \( z \) of the system sufficiently regular, then the normal speed to \( \Gamma(t) \) in \( x(t) \) at time \( t \) is
\[
v(x(t), t) = - \lim_{y \to x(t), y \notin \omega} \frac{\partial z}{\partial t}(y, t) \frac{\nabla z(y, t)}{||\nabla z(y, t)||}
\]
(4-24)

**Proof.** The system is assumed null-A-spreadable, then there exists a family of mappings \( F(\cdot, t) \) and a function \( \eta(x, t) \) such that
\[
z[F(x, t), t] = \eta(x, t)z_0(x) \ \forall x \in \Omega \ \text{and} \ \forall 0 < t < T
\]
(4-25)

We assume that \( \eta(x, t) \) is independent of time and space (we take \( \eta(x, t) = 1 \) and \( F(\cdot, t) \)) is a family of diffeomorphisms of \( \Omega \). We denote \( x = x(t). \)

Differentiating (4-25) with respect to time, we get \( \forall x \in \Omega \) and \( \forall 0 < t \leq T \)
\[
\frac{\partial F}{\partial t}(x, t)\nabla z[F(x, t), t] + \frac{\partial z}{\partial t}[F(x, t), t] = \frac{\partial \eta}{\partial t}(x, t)z_0(x)
\]
(4-26)

As we have assumed that \( \eta(x, t) \) is a constant, then (4-26)becomes
\[
\forall x \in \Omega \ \text{and} \ \forall 0 < t \leq T
\]
\[
\frac{\partial F}{\partial t}(x, t)\nabla z[F(x, t), t] + \frac{\partial z}{\partial t}[F(x, t), t] = 0
\]
(4-27)

As \( (F(\cdot, t)) \) is a family of diffeomorphisms of \( \Omega \), then for all \( x \in \Gamma(t) \), there exists \( x_0 \in \Gamma(0) \) such that \( F(x_0, t) = x \)and for \( x_0 \) in \( \Gamma(0) \), there exists a neighbourhood \( \mathcal{V}(x_0) \) of \( x_0 \)such that for all \( y \in \mathcal{V}(x_0) \) and \( y \notin \omega \), \( \nabla z[F(y, t), t] \neq 0 \).

Equation (4-27) is satisfied for all \( x \in \Omega \), then in particular for \( y \), then we can rewrite
\[
\frac{\partial F}{\partial t}(y, t) \frac{\nabla z[F(y, t), t]}{||\nabla z[F(y, t), t]||} + \frac{\partial z}{\partial t}[F(y, t), t] = 0
\]
(4-28)

\( \forall y \in \mathcal{V}(x_0) \) and \( y \notin \omega \)

or in the form
\[
\lim_{y \to x_0, y \notin \omega} \frac{\nabla z[F(y, t), t]}{||\nabla z[F(y, t), t]||} = - \frac{\partial z}{\partial t}(x_0, t)
\]
(4-29)

As
\[
\lim_{y \to x_0, y \notin \omega} \frac{\nabla z[F(y, t), t]}{||\nabla z[F(y, t), t]||} = \vec{n}(F(x_0, t), t)
\]
(4-30)

On the other hand, (4-29) gives
\[
\lim_{y \to x_0, y \notin \omega} \frac{\partial F}{\partial t}(y, t) \frac{\nabla z[F(y, t), t]}{||\nabla z[F(y, t), t]||} =
\]
(4-31)

which gives
\[
\frac{\partial F}{\partial t}(x_0, t) \vec{n}(F(x_0, t), t) = - \lim_{y \to x_0, y \notin \omega} \frac{\partial z}{\partial t}[F(y, t), t]
\]
(4-32)

As \( \vec{n}(F(x_0, t), t) \) is normal to \( \Gamma(t) \) at the point \( F(x_0, t) \), then
\[
v(F(x_0, t), t) = \frac{\partial F}{\partial t}(x_0, t)\vec{n}(F(x_0, t), t)
\]
(4-33)

we obtain finally
\[
v(F(x_0, t), t) = - \lim_{y \to x_0, y \notin \omega} \frac{\partial z}{\partial t}[F(y, t), t]
\]
(4-34)

And by replacing \( x = x(t) = F(x_0, t) \), we find the result.
5 Spray Control Problem

In this section we are interested by systems which are not $A$-spreadable but which can be made $A$-spreadable when they are excited by convenient controls. We recall that a spray (or $A$-spray) control is any control which make the system spreadable (or $A$-spreadable). In [5], we have studied the problem of determination of $A$-spray controls. For linear systems a characterization of such controls has been given.

As cited in [5], we can find an $A$-spray control if we choose, preliminarily, a form of geometric evolution of the domains $(\omega_t)$. This choice is not always simple. That is why we reconsider this problem in connection with the notion of evolving interfaces by controlling the front evolution.

5-a Example/Motivation

Example 5.1 Consider the system given by the equation

$$\begin{cases}
\frac{\partial z}{\partial t}(x,t) = b(x,t)z(x,t) + u(x,t) & t > 0, \ x > 0 \\
z(x,0) = z_0(x) = h(x+1) & x > 0
\end{cases}$$

(5-1)

where

$$h(x) = \begin{cases}
\exp\left(-\frac{1}{1-x}\right) & \text{if } 0 < x < 1 \\
0 & \text{if } 1 \leq x \leq 2 \\
\exp\left(\frac{1}{2-x}\right) & \text{if } x > 2
\end{cases}$$

and

$$b(x,t) = \begin{cases}
\frac{x - 2t - 1}{(x - t)^2(t + 1)^2} & \text{if } 0 < x < t \\
0 & \text{if } t \leq x \leq t + \frac{1}{t + 1} \\
-\frac{x - 2t - 1}{[1 - (x - t)(t + 1)]^2} & \text{if } x > t + \frac{1}{t + 1}
\end{cases}$$

Case where the system is autonomous

$$\begin{cases}
\frac{\partial z}{\partial t}(x,t) = b(x,t)z(x,t) & t > 0, \ x > 0 \\
z(x,0) = z_0(x)
\end{cases}$$

(5-2)

For $x > 0$ and $t > 0$, the solution is given by

$$z(x,t) = \begin{cases}
\exp\left(\frac{1}{(x - t)(t + 1)}\right) & \text{if } 0 < x < t \\
0 & \text{if } t \leq x \leq t + \frac{1}{t + 1} \\
\exp\left(\frac{1}{1 - (x - t)(t + 1)}\right) & \text{if } x > t + \frac{1}{t + 1}
\end{cases}$$

(5-3)

Consider the null-spreadability case with

$$\omega_t = \{ x \in \Omega \ | \ z(x,t) = 0 \} = [t, t + \frac{1}{t + 1}]$$

and $\omega_0 = [0, 1]$. In this case $\mu(\omega_t) = \frac{1}{1 + t}$, the system is then null-$A$-resorbable.

![Figure 2: Example of non $A$-spreadable autonomous system](image)

Controlled system

We consider now the system excited by the control $u_1$ given by

$$u_1(x,t) = \begin{cases}
\exp\left(\frac{2(x + 1) - x - 2t - 1}{(2x - t)^2 - (x - t)^2(t + 1)^2}\right) & \text{if } 0 < x < t/2 \\
0 & \text{if } t/2 \leq x \leq t + 1 \\
0 & \text{if } x > t + 1
\end{cases}$$

(5-4)

In this case the solution is given by

$$z_1(x,t) = \begin{cases}
\exp\left(\frac{1}{x - t/2}\right) & \text{if } 0 < x < t/2 \\
0 & \text{if } t/2 \leq x \leq t + 1 \\
\exp\left(\frac{1}{1 - x - t/2}\right) & \text{if } x > t + 1
\end{cases}$$

(5-5)
Let $\omega_t^u = \{ x \in \Omega : z(x, t, u) = 0 \}$ where $z(\cdot, t, u)$ is the state of the system excited by $u$. With $\omega_0 = [0, 1]$ we have $\omega_t^{u_1} = [\frac{t}{2}, t + 1]$. The system becomes null-$A$-spreadable under the effect of the control action $u_1(x, t)$. Let $\omega_t^u = \{ x \in \Omega : z(x, t, u) = 0 \}$ where $z(\cdot, t, u)$ is the state of the system excited by $u$. With $\omega_0 = [0, 1]$ we have $\omega_t^{u_1} = [\frac{t}{2}, t + 1]$. The system becomes null-$A$-spreadable under the effect of the control action $u_1(x, t)$.

This example proves that it is possible, by a convenient choice of a control to make a system $A$-spreadable. In this section we reconsider this problem in connection with the notion of evolving interfaces by considering a control $u$ which makes the system $A$-spreadable with a given speed of the front $\Gamma(t)$. In the following we only consider the case of null-$A$-spreadability.

5-b Problem statement

Consider the linear system

$$\begin{align*}
\dot{z} &= Az + Bu \\
z(t_0) &= z_0
\end{align*}$$

(5-6)

where $A$ generates a strongly continuous semi-group $(S(t))$ on the state space $Z = L^2(\Omega)$. $B$ is a linear operator from the control space $U$ to $Z$. $I = [t_0, T]$ is the time interval and $z_0 \in D(A)$ where $D(A)$ is the domain of $A$. The solution of (5-6) is given by

$$z(\cdot, t, u) = S(t - t_0)z_0(\cdot) + \int_{t_0}^{t} S(t - s)Bu(s) \, ds$$

(5-7)

We introduce the operator $H : \mathcal{V} \rightarrow L^2(\Omega \times I)$ with $\mathcal{V} = L^2(t_0, T; U)$, defined by

$$Hu(\cdot, t) = \int_{t_0}^{t} S(t - s)Bu(s) \, ds$$

(5-8)

then

$$z(\cdot, t, u) = S(t - t_0)z_0 + Hu(\cdot, t)$$

In the case of the null-spreadability, $\omega_t^u = \{ x \in \Omega : z(x, t, u) = 0 \}$, $t \in I$, $u \in \mathcal{V}$, we assume that $\mu(\omega_0) \neq 0$. Let $\Gamma(t) = \partial \omega_t$, be the front. Then the question is can we find a control which achieves the null-$A$-spreadability of the system? Moreover can the control lead to a given speed $v(x, t)$ of the front. We consider this problem as a spray control problem stated by

$$\begin{align*}
\text{Find a spray control } u \in \mathcal{V}\text{ such that } \\
\{ P \} &\text{ the front } \Gamma(t) \text{ evolves with a} \\
&\text{given speed } v(x, t)
\end{align*}$$

The problem consists to find a control $u$ which achieves the $A$-spreadability and with a given expansion speed of the front $\Gamma(t)$. This may be stated by the general formulation given in [5] using the family of mappings $(F(\cdot, t, s))_{t \geq s}$ of $\Omega$ adapted to the evolution of the system. We recall the result.

**Definition 5.2** A family of mappings $(F(\cdot, t, s))$ is said to be adaptable to the evolution of the system $(S)$ if there exists a control $u \in \mathcal{V}$ such that the family $(F(\cdot, t, s))$ is adapted to the evolution of $(S)$ excited by $u$. $u$ is said to be associated to $(F(\cdot, t, s))$.

A characterization of such family is given in [5].

**Proposition 5.3** A control $u$ is a spray control if it is associated to a family of mappings $(F(\cdot, t, s))$ which satisfies

$$\mu(F(\omega_s, t, s)) \geq \mu(\omega_s), \quad t \geq s \geq t_0$$

(5-9)

If $(F(\cdot, t, s))_{t \geq s}$ is a family of diffeomorphisms of $\Omega$, then condition (5-9) can be replaced by (2-5) or equivalently, (2-6) if $F(\omega_s, t, s) = \omega_t \forall s, t, \ t \geq s \geq t_0$.

And we have the following theorem [5]

**Theorem 5.4** If $(F(\cdot, t, s))_{t \geq s}$ is a given family of mappings of $\Omega$ adaptable to the evolution of the system $(S)$ and $u$ is a control in $U$, then $u$ is associated to $(F(\cdot, t, s))_{t \geq s}$ if there exists a function $\xi$ which satisfies

$$\Theta(\xi, u) = 0$$

(5-10)

where

$$\Theta(\xi, u) = \int_{t_0}^{T} z^2[F(x, s, t), s, u] \, ds - \xi(x, t)z(x, t, u)$$

Theorem 5.4 and Proposition 5.3 allows one to determine a spray control. For more details, see [5].

We consider the problem $(P)$. As cited in [5], we can find a spray control if we choose, preliminarily, a form of geometric evolution of the domains $(\omega_t)$ (the family of mappings $(F(\cdot, t, s))$). The solution of problem $(P)$ is carried out in two steps:

(i) find a family of mappings which describes the geometric evolution of domains $(\omega_t)$ in the case where the front
the result exists under suitable hypothesis. Theorem 5.4 gives the result.

The first step, i.e. the determination of the family of mappings \( (F(., t, s))_{t \geq s} \), through the vector speed, is hard because we do not have any explicit result. In [14] such family exists under suitable hypothesis.

5-c Application to one dimensional case

We consider here the system given by the equation (5-6) in the one dimensional case. Starting from \( w_0 = [0, 1] \), consider the spray control problem

\[
(P_0) \quad \begin{cases} 
\text{Find a null-spray control } u \in \mathcal{V} \text{ such that } \\
0 \text{ and } 1 \text{ evolves with given speeds}
\end{cases}
\]

Denote \( v(0, t) \) and \( v(1, t) \) the displacement speed of the point 0 and 1 respectively. Using Proposition 3.4, we obtain the result

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Autonomous system} & v_1(0, t) & v_1(1, t) & \omega_1^{(u)} \\
\hline
\text{control } u_1(x, t) & 1 / 2 & 1 & \left[ t, t + \frac{1}{t+1} \right] \\
\hline
\text{control } u_2(x, t) & 1 & 2 & \left[ t, 2t + 1 \right] \\
\hline
\text{control } u_3(x, t) & 1 / 2 & 3 & \left[ \frac{3}{2}, 3t + 1 \right] \\
\hline
\end{array}
\]

Proposition 5.5 The problem \((P_0)\) has a solution if the family of mappings defined by \( F(x, t, t_0) = a(t)x + b(t) \), with

\[
\begin{cases}
\text{i) } \quad b'(t) = v(0,0) \\
\text{ii) } \quad a'(t) + b'(t) = v(1,0)
\end{cases}
\]
and the initial conditions

\[
b(0) = 0 \quad \text{and} \quad a(0) = 1
\]
is adaptable to \( (S) \).

The result derives immediately from theorem 5.4.

5-d Example

We consider the example 5.1

\[
\begin{cases}
\frac{\partial z}{\partial t}(x, t) = b(x, t)z(x, t) + u(x, t) & t > 0, x > 0 \\
z(x, 0) = z_0(x) = h(x+1) & x > 0
\end{cases}
\]
and the problem \((P_0)\). Proposition 5.5 gives a family of mappings \( (F(., t, s))_{t \geq s} \) which is adaptable to the evolution of the system. Theorem 5.4 gives a control solution of problem \((P_0)\). Recall that a family of mappings \( (F(., t, s))_{t \geq s} \) is adapted to the evolution of the system \((S)\) if there exists a function \( \eta(x, t, s) \) which satisfies the equation

\[
z[F(x, t, s), t] = \eta(x, t, s)z(x, s)
\]

This equation proves that the control \( u \) depends of course on the choice of the family of mappings \( (F(., t, s))_{t \geq s} \), but also of the function \( \eta(x, t, s) \) (which is generally unknown). Taking \( \eta(x, t, t_0) = 1 \) (independent of time and space), we obtain an explicit expression of \( u \). Indeed, let

\[
Z(x, t, u) = z_0[F^{-1}(x, t, t_0)] = Z(x, t)
\]

where \( F^{-1}(x, t, t_0) \) is the converse function of \( F(x, t, t_0) \) with respect to \( x \), then from (5-12), we get an expression of the control \( u \).

\[
u(x, t) = \frac{\partial Z}{\partial x}(x, t) - b(x, t)Z(x, t) \quad t > 0, x > 0.
\]

which is a solution of problem \((P_0)\). Notice that it is not unique. Thus we have the algorithm

1. \( v(0, 0) \) and \( v(1, 0) \) are given,
2. find the family of mappings \( (F(., t, s))_{t \geq s} \), through the proposition 5.5,
3. replace \( F(x, t, t_0) \) in (5-14) (with \( \eta(x, t, s) = 1 \)),
4. replace \( Z(x, t) \) in (5-15).
For this example, we give in the following table the results for different values of speed $v(0, t)$ and $v(1, t)$ and different controls.

Where, for $i = 1, 2$ and $3$,

$$u_i(x, t) = \frac{\partial Z_i(x, t)}{\partial t} - b(x, t)Z_i(x, t) \quad t > 0, \quad x > 0$$

$$Z_i(x, t) = z_0[F_i^{-1}(x, t, 0)] \quad \text{and} \quad F_i(x, t, 0) = a_i(t)x + b_i(t)$$

where

$$\left\{ \begin{array}{l}
    b'_i(t) = v_i(0, t) \\
    a'_i(t) + b'_i(t) = v_i(1, t)
\end{array} \right. \quad (5-16)$$

and the initial conditions

$$b_i(0) = 0 \quad \text{and} \quad a_i(0) = 1.$$

This example proves that it is possible, by a good choice of a control $u$, to make a system $A$-spreadable and we can also control the spread velocity. This depends also on the speed of the front (boundary of $\omega_t$). In the particular situation of example 5.1, the equation 5-15 (for a given $\eta(x, t, t_0)$) gives an explicit form of the control $u$ because the operator $B = 1$.

But for the general cases (or if $\eta(x, t, t_0)$ is unknown), Theorem 5.4 and Proposition 5.5 gives an algorithm for computing a solution of the problem.

6 Conclusion

The concept of spreadability introduced by El Jai and al. in [8, 9] offers a framework which extends most of spatio-temporal problems including a front motion. In this paper we have tried to show how they are related to the spreadability, the travelling wavefront problems and evolving interfaces ones. The spray control problem has also been studied. We have shown that it is possible, for linear systems and under some hypothesis, to determinate a spray control which achieves the spreadability with a given expansion speed of the front. Various examples illustrate the possible connections. The description of these phenomena by partial differential equations offers a mathematical framework which remains very hard for practical minded modelers. That is why we try to describe these phenomena by the above approaches. However the numerical implementation remains a difficult problem and an interesting alternative is that of cellular automata which offers a simpler tool for numerical simulations as it is under consideration in [10].

References


