On the Invertibility of SISO Systems with Singularity in the Inverse Map

R. Stojic, C. Chevallereau
IRCyN, UMR C.N.R.S. 6597,
1 rue de la Noé, BP 92101,
44321 Nantes Cedex 3, France.
{R.Stojic,C.Chevallereau}@ircyn.ec-nantes.fr

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Abstract

In this paper attempt is made to treat invertibility of class of systems with singular inverse in extended state space, possibly of infinite dimension. Using no approximation, necessary conditions for invertibility are formulated, as well as sufficient conditions for invertibility within finite-dimensional state space framework, and also conditions under which the limit process (dimension of state $\rightarrow \infty$) annihilates influence of singularity. The proposed algebraic procedure for the constructing of output inverse may be also applied to systems having inverse map close to singular.

1 Introduction

The study is motivated by inverse problems (e.g. in the aircraft and biped dynamics), characterized by phenomena of singularity or that one close to singularity.

Inverse model of controlled object, or plant, is often exploited in practice. Static inverse is often implemented online within control loop, e.g. inverse kinematic scheme in robotics. Inverse model based on complete nonlinear model of plant dynamics allows analysis of even nonstationary regimes and full insight based on available model description. It can also be met in on-line implementation in flight control or as a popular computed torque in robotics.

For these reasons inverse system dynamics is the subject of study of control theoreticians, starting with famous [1] for linear models and generalization [3] for nonlinear systems, see also [2],[4]. Additional reason is that system invertibility is close to study of system structure and existence of feedback linearizable transformations [5]. Namely, flat system is invertible, but converse fails, implying that invertibility is wider concept.

In a wide class of systems, satisfying (sufficient) conditions for invertibility, well defined procedures for system inverse construction exist. However, if the inversion procedure leads to description with algebraic equations which does not allow to extract higher order derivative of input in the region of interest (ODE with singularity), the existing procedures lacks efficiency. The similar phenomena arises when higher order of input derivative is multiplied by nonzero but small term inducing "highly unstable" behaviour of solutions. These problems, to the knowledge of authors, are steel without complete theoretical treatment.

This problem is studied in the context of output tracking task [6],[7], or feedback linearization [8]. In [8] an approximate technique is proposed, in which so called nonminimal phase terms are neglected using sufficient conditions (existence of equilibrium state) and intuition.

2 Problem Statement

Consider a SISO system given by

$$\dot{x} = \tilde{f}(x,u) \quad y = h(x)$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ the input, $y \in \mathbb{R}$ the output, and $\tilde{f} \in C^\infty, h \in C^\infty$. We assume that (1) is by an equivalence transformation reduced to the form

$$y^{(k)} + \chi(y^{(n-1)}, ..., y, u^{(k)}, u^{(k-1)}, ..., u) = 0$$  \hspace{1cm} (2)

where $0 < k \leq n, \chi \in C^\infty$. Let $t \in T = (t_0, t_k)$, be the time and $\mathcal{U} = \{u(t) \mid T \mapsto D_u \subset \mathbb{R}\}$, $\mathcal{U} \subset C^\infty$ is a class of input functions with $D_u$ connected, open and bounded. According to the theory of ODE, $\forall u(t) \in \mathcal{U}, \exists y(t) \in C^\infty$, satisfying (1) and depending on initial conditions. Thus we may associate mapping $S$ to eq. (1) i.e. (2):

$$S : \mathcal{U} \mapsto \mathcal{Y} \quad \mathcal{Y} \subset C^\infty$$ \hspace{1cm} (3)

where $y(t) \in \mathcal{D}_y, \forall t \in T, \forall u(t) \in \mathcal{U}$. $\mathcal{D}_y$ is also open and bounded. The inverse mapping

$$S^{-1} : \mathcal{Y} \mapsto \mathcal{U}$$ \hspace{1cm} (4)

associates $u(t) \in \mathcal{U}$ to given $y(t) \in \mathcal{Y}$, such that (2) is satisfied identically.

If the output $y(t) \in C^\infty$ is given, we may state the following questions:

a) Is $y(t) \in C^\infty \Rightarrow u(t) = S^{-1}(y(t)) \in C^\infty$, i.e. does given smooth output implies existence of smooth input?
b) If \( y(t) = S(u(t)) \), i.e. if \( y(t) \) is generated by smooth input \( u(t) \), is the inverse unique?

c) What are conditions for existence of unique inverse and how to construct it, if exists?

If eq. (2) uniquely determines higher order derivative of input \( u^{(k)} \) (i.e. \( \frac{\partial \chi}{\partial u^{(k)}} \neq 0 \)), then inverse mapping \( S^{-1} \) is regular and possesses similar properties like \( S \). This case is extensively studied [3],[2],[4] and will not be considered here.

Thus let us assume that the derivative \( \frac{\partial \chi}{\partial u^{(k)}} \) may take zero value for some values of arguments.

The answers to the first two questions are given by the following note, illustrated by the example.

**Note 1**

In the singular case the inverse mapping \( S^{-1} \) lacks continuity and conditions for uniqueness and existence of solutions of corresponding ODE are not satisfied. So called left and right inverse are not equivalent. Left inverse always exists (by definition), but may not be unique. Right inverse of given \( y(t) \in C^\infty \) may not exist in the class of smooth inputs.

**Example 1.**

For the system

\[
\dot{y} = u - u\dot{u}
\]

\( u \in R, y \in R, c = const. \)

\[
y = c \quad \Rightarrow \quad u = S^{-1}(c) = \begin{cases} 0 \\
t \end{cases} \]

Thus, the inverse of output is not unique.

**Example 2.**

Consider the system

\[
\dot{y} = u - y\dot{u}
\]

\( u \in R, y \in R. \) Let \( c \) be arbitrary constant. Then

\[
y = c \quad \Rightarrow \quad u = S^{-1}(c) = \begin{cases} u_0e^{\frac{c}{t}} \\
0 \quad c \neq 0 \\
0 \quad c = 0 \end{cases}
\]

where \( u_0 = const \) is arbitrary.

\[
y = ct \quad \Rightarrow \quad u = S^{-1}(ct) = \begin{cases} c + u_0t^{\frac{c}{t}} \\
0 \quad c \neq 0 \\
0 \quad c = 0 \end{cases}
\]

In the second case, if \( c < 0 \) the inverse is not defined for \( t = 0 \).

To study the singular case, we introduce the following notation

\[
Y_n = [y, \dot{y}, ..., y^{(n)}]^T \\
U_k = [u, \dot{u}, ..., u^{(k)}]^T
\]

i.e., each function of time is denoted by a small letter (e.g. \( \gamma(t) \)). The vector made up using first \( i \) derivatives of this function with respect to time, is denoted by indexed capital letter (i.e. \( \Gamma_i(t) \)).

The eq. (2) may be written in the form

\[
f(Y_n, U_{k-1}) + e(Y_{n-1}, U_{k-1})u^{(k)} = 0
\]

where \( f, e \in C^\infty \).

Let \( (Y_{n-1}, U_{k-1}) \in D \), where \( D \subset R^{n+k} \) is open and bounded. The manifold

\[
E = \{(Y_{n-1}, U_{k-1})| e(Y_{n-1}, U_{k-1}) = 0 \}\subset D
\]

is called the singularity manifold. Here, \( E \) is assumed to be nonempty, nontrivial connected and simple (i.e. equivalent to a single hyper-surface in \( R^{n+k} \)).

### 3 Conditions for existence of the inverse mapping

Following the idea of equal role of the output and input variables [10], the pair \( (y(t), u(t)) \) and \( (Y_{n-1}(t), U_{k-1}(t)) \) also satisfying (2) will be called trajectory.

For chosen \( u(t) \in U \) and arbitrary initial condition \( (Y_{n-1}(0), U_{k-1}(0)) \) \( \in D \), if the corresponding trajectory over its interval of existence does not have common points with \( E \), then a new region \( D_1 \subset D \), \( D_1 \cap E = \phi \) may be chosen such that mapping \( S^{-1} \) is regular in \( D_1 \).

Thus, let us assume that \( D \) is chosen in such a way that all trajectories in \( D \) have at least one common point with \( E \). Choose this common point as initial condition, i.e. assume that \( (Y_{n-1}(0), U_{k-1}(0)) \in E \) is arbitrary but fixed.

The manifold \( E \) may be invariant with respect to trajectories i.e.

\[
(Y_{n-1}(t), U_{k-1}(t)) \in E, \forall t \in T. \tag{7}
\]

The necessary condition for this are

\[
e(Y_{n-1}(t), U_{k-1}(t)) = 0 \\
f(Y_{n-1}(t), U_{k-1}(t)) = 0 \quad \forall t \in T \tag{8}
\]

If the above equations are consistent, then this is sufficient condition for existence of trajectories lying on \( E \).

In the example 1., we have \( E = \{(y, u) | u = 0 \} \). If \( u = 0 \), differential equation reduces to \( \dot{y} = 0 \) and thus, all trajectories \( (y, u) = (c, 0) \) lie on \( E \).

In the following, for study of trajectories passing trough \( E \), in order to simplify derivations in the proofs of propositions, we will restrict to case \( k = 1 \) (inverse mapping is represented by first order singular ODE):

\[
f(Y_n, u) + e(Y_{n-1}, u)\dot{u} = 0
\]

Assuming \( (Y_n, u) \) to be function of time formal differentiation of (9) gives the system:

\[
f^{(0)} + e^{(0)}\dot{u} = 0 \\
f^{(1)} + e^{(1)}\ddot{u} + e^{(0)}\dot{u} = 0 \\
... \\
f^{(0)} + e^{(1)}\dddot{u} + i\epsilon^{(2)}\ddot{u} + ... + i\epsilon^{(1)}u + e^{(0)}u^{(1)} + e^{(0)}u^{(1)} = 0
\]

\( i \epsilon \) is constant.
which, using the introduced notation may be written in the compact form (see Appendix for details) as

$$ F_i(Y_{n+i}, U_i) + e(Y_{n-1}, u)U_i = 0 \quad (11) $$

We may call (11) as associated system of equations to system (9), or simply the associated system.

The following proposition, whose proof is trivial, may be formulated:

**Proposition 1**

Systems (9) and (11) are equivalent.

A necessary conditions for the existence of inverse mapping are formulated in the following proposition.

**Proposition 2**

If pair $(y(t), u(t))$, satisfying (9) exists and $(Y_{n-1}(0), u(0)) \in E$. Then for every $i = 0, 1, ...$

$$ F_i(Y_{n+i}(0), U_i(0)) = 0 \quad (12) $$

**Proof:** Let $(y(t), u(t))$ be pair satisfying (9) identically.

Then, the equation (12) follows from Proposition 1 and relation (11).

Assuming that necessary conditions are satisfied, a sufficient conditions for existence of control $u(t) = S_i^{-1}(y(t))$, are defined in the following.

To the following system of algebraic equations

$$ F_i(Y_{n+i}(t), U_i) + e(Y_{n-1}(t), U_i)U_i = 0 \quad (13) $$

let us associate the mapping $S_i$ defined in $D$ such that

$$ y(t) = S_i(u(t)) \quad (14) $$

if and only if pair $(y(t), u(t))$ satisfies (13) identically.

In the following it will be shown that mapping $S$ may be approximated by $S_i$, and that difference may be made arbitrary small when $i \to \infty$.

Due to the previously introduced assumptions, all partial derivatives of $e, f$ are bounded in $D$, so denoting $f_u(Y, u) = \frac{\partial f}{\partial u}$ we may formulate the following proposition.

**Proposition 3**

Let $(Y_{n-1}(0), u(0)) \in E$ and

$$ |f_u(Y, u)| \geq m_0 > 0, \quad \forall(u, Y) \in D. $$

Then, for every $i > 0$ a nonempty neighborhood $V_i \subset R^{(n+2i+2)}$ of initial condition $(Y_{n+i}(0), U_i(0))$ exists such that for $\forall(Y_{n+i}, U_i) \in V_i$ the mapping $S_i$ is continuous and invertible.

The proof is based on implicit function theorem (using the fact that the rank of the Jacobian $\frac{\partial(F_i + eU_i)}{\partial(Y_{n+i}, U_i)} = i + 1$ on $E$), which imply the existence of neighborhood $V_i$ in which rank of the Jacobian is constant (see appendix).

To formulate following proposition, observe the jacobian $\frac{\partial(F_i + eU_i)}{\partial U_i}$. On $E$, the Jacobian takes lower triangular form (see appendix) and its determinant is

$$ \Delta_i(Y_n, U_i) = f_u \prod_{j=1}^{i} (f_u + (j + 1)\epsilon_u + j\epsilon_yY_{n-1}) \quad (15) $$

where $f_u, \epsilon_y, \epsilon_u$ are partial derivatives of $f, e$ with respect to $Y_{n-1}$ and $u$.

**Proposition 4**

Let $\epsilon > 0$ and $u(t) \in U$ be arbitrary. Let $|f_u(Y, u)| \geq m_0 > 0$, $\forall(Y_{n-1}, u) \in D$. Then a number $i$ and a nonempty neighborhood $V_i \subset R^{(n+2)}$ of $(Y_{n-1}(0), u(0)) \in E$ exist such that

$$ \|u(t) - S_i^{-1}(S(u(t)))\| < \epsilon \quad (16) $$

$\forall t, such that $(u(t), S(u(t))) \in V_i$.

if the expression

$$ \frac{e^i(Y_{n-1}, u)}{\Delta_i(Y_n, U_i)} u^{(i+1)} \quad (17) $$

may be made arbitrary small in $V_i$ by choosing sufficiently large $i$.

**Proof** is based on Proposition 3., i.e. on existence of $V_i$ in which $S_i$ is continuous, and invertible. For details see appendix.

If the assumptions of Proposition 4 are satisfied, then the inverse is unique.

**Note 2**

If $U = P_m(t)$ - class of polynomials with finite degree $m$, or $U = \Omega_{\omega_0}$ - class of periodic inputs with bounded frequency, then conditions of the Proposition 4. are satisfied for any $m$ or any input with $\omega < \omega_0$.

**Note 3**

Relation (11) may be interpreted as algebraic mapping from $R^{n+i+2}$ to $R^{n+i+1}$

$$ (Y_{n-1}, U_{i+1}) \leftrightarrow Y_{n+i} $$

and (13) as algebraic mapping from $R^{n+i+1}$ to $R^{n+i+1}$

$$ (Y_{n-1}, U_i) \leftrightarrow Y_{n+i} \quad (18) $$

In the second case one-to-one correspondence may be established.

Let us assume that in (1) the function $h(x)$ is chosen is such a way that for $y = h(x)$ one obtains the form (2) with lowest possible $k$. If $k = 0$ system is flat [10], and may not have singular inverse map.

However, for the case considered in the paper (having in mind Note 2) relation (18) may be interpreted as transformation which brings (9) into form

$$ y^{(n+i)} = v $$
where \( v \) is auxiliary input, and consider above form, in restricted sense, as flat approximation to the system (9).

**Note 4**

The proposition 4, given here in the context of singular inverse map, allows a reformulation based on assumption of existence of neighborhood \( \mathcal{V} \subset \mathbb{R}^{n+2} \) in which expression (17) may be made arbitrary small. Thus, condition \( \varepsilon = 0 \) is not necessary for proposition 4 and the proposed inversion procedure is also applicable to systems with nonsingular inverse map.

The following example with regular map is given to indicate a possible extensions of presented invertibility procedure, and also (since the singularity phenomena in the inverse map is often called as “nonminimum phase effects”) to indicate relation with linear nonminimum phase systems.

**Example 3.**

Consider the system with unstable inverse map

\[
y' - y = u - \varepsilon u
\]

(19)

where \( y, u \in \mathbb{R}, \varepsilon > 0 \) is small number. For all \( y \in C^\infty \) all solutions of inverse are unstable. The proposed inversion procedure gives

\[
u = \sum_{j=0}^{\infty} (y^{(j+1)} - y^{(j)}) \varepsilon^i
\]

(20)

If \( y \) is bounded and sufficiently smooth (i.e. series is uniformly convergent), then \( u \) is also bounded and smooth, representing the only solution with such properties. Namely, it is separatrix of two branches of unstable solutions, all of which tend to either \( +\infty \) or \( -\infty \) when \( t \to \infty \).

Taking \( i = \infty \), Laplace transform of the above relation gives

\[
u(s) = y(s)(s - 1) \sum_{j=0}^{\infty} (\varepsilon s)^i = y(s) \frac{s - 1}{1 - \varepsilon s}
\]

which is exactly the inverse transfer function of (19). The only difference is that (20) is stable for any \( i \), and may be applied if \( |s| < 1 \) i.e. within the region of uniform convergence of the series.

**4 An analysis of biped dynamics**

Biped, in a certain dynamic regimes, is an example of system with singular inverse map.

By suitable choice of generalized coordinates \( p, q \), two degree of freedom mechanical system equations may be represented in the form [9],[11]:

\[
J(q)\ddot{p} = e(\dot{p}, q)\dot{q} + x_{cm}(p, q)
\]

\[
\dot{q} = v
\]

\[
J(q) = l + \cos 2q \quad e(\dot{p}, q) = 2\dot{p}\sin 2q
\]

(21)

\[x_{cm}(p, q) = g(l\sin(p + q) + \sin(p - q))\]

where \( J(q) \) is the instantaneous moment of inertia, \( x_{cm} \) center of mass position and \( l, q, (l < 1) \) are constants. In the walking \( p, q \) are bounded, i.e. \( p \in (-p_m, p_m), q \in (-q_m, q_m) \), for some constants \( p_m, q_m \). In the analysis of inverse map the second of the eqs. (21) may be omitted (if \( q(l) \) is known \( \Rightarrow \dot{q}(l) \)). Thus, consider \( q \) as system input.

Biped singularity manifold

\[
E = \{(q, \dot{p})|2\dot{p}\sin 2q = 0\}
\]

is the union of hypersurfaces \( q = 0 \) and \( \dot{p} = 0 \) in the state space, and evidently, \( E \) is not simple.

Consider the singularity submanifold \( q = 0 \).

A necessary condition (12) for the existence of trajectories passing through this submanifold gives:

\[
e(q, \dot{p}) = 0, \quad q = 0 \Rightarrow J(q, \dot{p})p = x_{cm}(p, 0)
\]

i.e. output \( p \) must satisfy the above equation.

Analyze of sufficient conditions (Prop. 4) gives:

\[
e_p = 0 \quad e_{\dot{p}} = 2\sin(2q) \quad e_q = 4\dot{p}\cos(2q)
\]

\[
f_q = -2\dot{p}\sin 2q - gl\cos(p + q) + g\cos(p - q)
\]

(lover index denotes partial derivation). On \( E \)

\[
e_p = 0 \quad e_{\dot{p}} = 0 \quad e_q = 4\dot{p}
\]

\[
f_q = g(1 - l)\cos p
\]

Choosing \( i < \infty \) and \( m_0 > 0 \) such that \( \cos p_m = m_0, \quad p \in (-p_m, p_m) \) assures that \( f_q > 0 \), and therefore, the jacobian

\[
\Delta = \sum_{j=1}^{i} (f_q + 4(j + 1)\dot{p}\dot{q})
\]

is nonsingular for \( \dot{p}, \dot{q} \) sufficiently small.

If inverse exists (i.e. if \( p(t) \) is ‘feasible’), input \( q \) may be obtained by solving of algebraic system (13):

\[
J(q)\ddot{p} = e(\dot{p}, q)\dot{q} + x_{cm}(p, q)
\]

\[
J(q)p^{(3)} = e(\dot{p}, q)\dot{q} + x_q(p, q)\dot{q} + \varphi_1(p, \dot{p}, \ddot{p}, q, \dot{q})
\]

\[
\ldots
\]

\[
J(q)p^{(i+2)} = e(\dot{p}, q)q^{(i+1)} + x_q(p, q)q^{(i)} + \varphi_i(p, \dot{p}, \ldots, p^{(i+1)}, q, \dot{q}, \ldots, q^{(i)})
\]

where \( \varphi_1, \ldots, \varphi_i \) denotes the rest of terms in the equations, obtained by successive differentiation of preceding equation.

Note that in applications may be sufficient to extract \( q, \dot{q}, \ddot{q} \) (i.e. to have \( i \leq 3 \)).
5 An example of aircraft inverse

Inverse map of conventional aircraft is regular, but characterized by so called "slightly nonminimal phase effect", due to presence of positive zero in linearized model of an aircraft dynamics. Thus, it may be considered close to singular.

Using so called flat earth approximation, quasistationary aerodynamic flow field and atmosphere at rest assumptions [12],[13], aircraft longitudinal dynamics may be described by the equations

\[
\begin{align*}
m\dot{V} &= F - QS(c_{x\theta} - \kappa c_{x}^2) - mg\sin(\theta - \alpha) \\
m\dot{V}(q - \dot{\alpha}) &= QS\dot{c}_{z} - mg\cos(\theta - \alpha) \\
J\dot{q} &= Qsle_m \\
\dot{\theta} &= q \\
a_n &= m\dot{V}(q - \dot{\alpha}) \\
c_z &= c_{z\theta} + c_{z\alpha}\alpha + c_{zq}q + c_{z\delta}\delta_m \\
c_m &= c_{m0} + c_{m\alpha}\alpha + c_{mq}\dot{\alpha} + c_{mq}q + c_{m\delta}\delta_m
\end{align*}
\]

where \( F, \delta \) are inputs, \( \theta, q, \alpha \) states \( V, \dot{V}, a_n \) outputs and all other are constants. Since \( \dot{V}, a_n \) are tangential and normal accelerations of the flight path, they are considered as known function of time, as well as velocity \( V \) and dynamic pressure \( Q \).

The thrust \( F \) may be obtained from the first equation, provided that solution of the rest of the equations is known and this equation will not be considered further.

Due to limited space and scope of the paper, we here show the way how the instability in the inverse map may be resolved (for more complete treatment see [13] where six DOF inverse model is presented). Since instability arises in dynamics characterized by fastest mode, we exploit so called "short period approximation" (i.e. \( V = \text{const} \) and LTI model) [12]:

\[
\begin{align*}
V\dot{\alpha} &= Z_{\alpha}\alpha + Vq + Z_{\delta}\delta \\
\dot{q} &= M_{\alpha}\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_{q}q + M_{\delta}\delta \\
a_n &= V(q - \dot{\alpha}) = Z_{\alpha}\alpha + Z_{\delta}\delta
\end{align*}
\]

to describe two DOF motion of the aircraft. Using the procedure presented in the paper, we add the equations

\[
\begin{align*}
\dot{a}_n &= \frac{\dot{a}_n}{Z_{\alpha}} \\
\dot{\delta} &= 0
\end{align*}
\]

As a consequence we have \( \dot{\alpha} = \frac{\dot{a}_n}{Z_{\alpha}} \) which implies

\[
\dot{a}_n = V \left( \dot{q} - \frac{\dot{a}_n}{Z_{\alpha}} \right)
\]

Obtained system of six algebraic equations may be solved with respect to variables \( \dot{\alpha}, \alpha, \dot{q}, q, \delta, \delta \) as functions of \( a_n, \dot{a}_n, \dot{a}_n \). It enables \( F \) to be calculated and thus represents stable aircraft inverse map.

Using data [12] for Boeing 747 (altitude 40000 ft, Mach = 0.9): \( V = 516, Z_{\alpha} = -339.0, \)

\[
Z_{\delta} = -18.341, M_{\alpha} = -1.615, M_{\dot{\alpha}} = -0.1425, M_{q} = -0.4038 M_{\delta} = -1.2124, \] one may obtain "exact" (term exact is questionable having in mind complexity and uncertainties in the aerodynamic flow field) inverse:

\[
\frac{\delta}{a_n}(s) = \frac{-18.341s^2 + 1.203s + 1.882}{(s + 4.841)(s - 4.295)}
\]

Unstable term \( (s - 4.295) \) may be replaced by stable series approximation (see example 3) convergent for \( |s| < 4.295 \). It may be easily verified that aircraft bandwidth is more narrow, i.e. this alternative inverse is valid in the frequency range of interest.

The same procedure may be applied to complete aircraft model (omitting assumption \( V = \text{const} \)), resulting in more accurate and more complex inverse whose stability is implied by stability of linear inverse.

6 Conclusion

It shown that analysing system properties on the singularity manifold may give insight into input-output relationship for the case when inverse map is singular. Necessary as well as sufficient conditions for existence of output inverse are formulated. It is also noted possibility of invariance of singularity manifold with respect to class of system trajectories.

Proposed algebraic input-output mapping exhibits convergence property to inverse map of given system in the neighborhood of singularity manifold. Outside, solutions may be extended by conventional ODE approach.

Note that invertibility is guaranteed for sufficiently slow inputs, e.g. for periodic inputs with bounded frequency. Also, if \( U \) is class of time polynomials of finite degree, than, input may be reconstructed within finite dimensional state space.

The technique may also be extended to systems with regular inverse map, characterized by nonzero but small term multiplying higher order input derivative, avoiding thus phenomena of chaos.

References


and in a similar way:

\[
\begin{align*}
  f^{(0)} &= f(Y_n, u) \\
  f^{(1)} &= f_Y(Y_n, u)\dot{Y}_n + f_u(Y_n, u)\dot{u} \\
  &\quad + \phi_i(Y_{n+i-1}, U_{i-1}) \\
  f^{(i)} &= f_Y(Y_n, u)Y_n^{(i)} + f_u(Y_n, u)u^{(i)} \\
  &\quad + \phi_i(Y_{n+i-1}, U_{i-1}) \\
  \end{align*}
\]

where \( \psi_i, \phi_i \) represent the rest of terms in the corresponding expressions, obtained by derivation with respect to time. Denoting by

\[
g(Y_n, U_1) = f(Y_n, u) + e(Y_{n-1}, u)\dot{u}
\]

equation (9) may be written as

\[
g(Y_n, U_1) = 0
\]

Introducing

\[
G_i = [g^{(0)}, g^{(1)}, \ldots, g^{(i)}]^T
\]

the associated system (10) takes the form

\[
G_i(Y_{n+i}, U_{i+1}) = 0
\]

From eqs. (10), it follows that \( G_i \) is linear with respect to \( e \). Introducing

\[
F_i(Y_{n+i}, U_i) = G_i(Y_{n+i}, U_{i+1}) - e(Y_{n-1}, u)\dot{U}_i
\]

we obtain the associated system in the form (11).

### A2 Proof of the Proposition 3

Let \( u \in \mathcal{U} \) and \( y(t) = S_i(u(t)) \) be given. Consider (13) as implicit definition of \( U_i \) in the form

\[
G_i(Y_{n+i}, U_{i+1}) = 0 \quad u^{(i+1)} = 0
\]

which may be rewritten as

\[
\tilde{G}_i(Y_{n+i}, U_{i+1}) = 0
\]

For \( 1 \leq j \leq i \) the partial differentiation of (22,23) gives

\[
\begin{align*}
  \frac{\partial f^{(j)}}{\partial u^{(j)}} &= f_u; \\
  \frac{\partial e^{(j)}}{\partial u^{(j+k)}} &= e_u; \\
  \end{align*}
\]

while from (10) it follows

\[
\begin{align*}
  \frac{\partial g^{(j)}}{\partial u^{(j+1)}} &= f_u; \\
  \frac{\partial g^{(j)}}{\partial u^{(j+k)}} &= e_u; \\
  \end{align*}
\]

We see that for points on \( E \) (if \( e = 0 \) is satisfied) jacobian

\[
D_i(Y_{n+i}, U_i) = \frac{\partial G_i}{\partial U_i}
\]

is diagonal and its determinant \( \Delta_i = det(D(Y_{n+i}, U_i)) \) is

\[
e(u, Y_{n-1}) = 0 \implies
\]
\[ \Delta_i = f_u \prod_{j=1}^{i}(f_u + (j+1)\epsilon_u \dot{u} + j\epsilon_y \dot{Y}_{n-1}) \]  

(33)

Since, by assumption is \( f_u \neq 0 \), it is evident that for sufficiently small \( \dot{u}, \dot{Y}_{n-1} \), the jacobian is nonsingular on \( E \). Since determinant is continuous function of its arguments, using proposition 1, this implies existence of neighborhood \( \nu \) of \( (Y_{n-1}, u) \in E \) in which jacobian is nonsingular.

A3 Proof of Proposition 4

Observing eq. (30), by convention denote

\[ \tilde{G}_i = [\tilde{g}^{(0)}, \tilde{g}^{(1)}, \ldots, \tilde{g}^{(i)}]^T \]  

(34)

Comparing expressions (26,27,29,30), it easy to see that

\[ g^{(j)} = \tilde{g}^{(j)}, \quad j = 0, 1, \ldots, i - 1 \]  

(35)

\[ g^{(i)} = \tilde{g}^{(i)} + \epsilon u^{(i+1)} \]  

(36)

Let \( u \in U \) and \( y(t) = S(u(t)) \) be given. Choose \( i > 0 \). By Proposition 3 exists neighborhood \( \nu_i \) such that \( S_i \) is invertible. Let

\[ u^* = S_i(y) \quad u = S(y) \]  

(37)

If

\[ U_i^* = [u^*, \dot{u}^*, \ldots, u^{(j)}]^T \]

\[ U_i = [u, \dot{u}, \ldots, u^{(i)}]^T \]

then, by definition, the following systems are satisfied

\[ \tilde{G}_i(Y_{n+i}, U_i^*) = 0 \quad \tilde{G}_i(Y_{n+i}, U_i) = 0 \]  

(38)

Having in mind (35,36) we obtain

\[ \tilde{G}_i(Y_{n+i}, U_i^*) = 0 \]  

(39)

\[ \tilde{G}_i(Y_{n+i}, U_i) + e b_i u^{(i+1)} = 0 \]  

(40)

where

\[ b_i = [0 \ 0 \ldots \ 0 \ 1]^T \]

is vector having all but last entry equal to zero.

Subtracting above equations, we obtain

\[ \tilde{G}_i(Y_{n+i}, U_i^*) - \tilde{G}_i(Y_{n+i}, U_i) = e b_i u^{(i+1)} \]  

(41)

Since jacobian \( \frac{\partial \tilde{G}_i}{\partial U_i} \) is nonsingular in suitably chosen \( \nu_i \), we may write

\[ \frac{\partial \tilde{G}_i}{\partial U_i}(U_i^* - U_i) + \theta_i = e b_i u^{(j+1)} \]  

(42)

where

\[ \| \theta_i \| \rightarrow 0 \quad \text{when} \quad U_i^* \rightarrow U_i \]

Observing relations (22) shows that entries on upper subdiagonal of jacobian \( \frac{\partial \tilde{G}_i}{\partial U_i} \) are all equal to \( \epsilon \), and entries on other upper subdiagonals are zero. Applying Cramer’s rule to the system (42) gives solution

\[ (u^* - u) = \frac{e_j}{\Delta_i} u^{(j+1)} - D_1 \theta_i \]  

(43)

where \( \Delta_i \) is jacobian determinant and \( D_1 \) is first row of the inverse jacobian.

Let \( \epsilon \) be given. Choosing \( \eta \) such that

\[ \| U_i^* - U_i \| < \eta \quad \Rightarrow \quad \| D_1 \theta_i \| < \epsilon_1 \]  

(44)

and number \( j_0 \) such that

\[ i > j_0 \quad \Rightarrow \quad \| e_j u^{(j+1)} \| < \epsilon_2 \]  

(45)

implies that

\[ |u - u^*| < \epsilon = \epsilon_1 + \epsilon_2 \]  

(46)

i.e. \( \| u(t) - S_i^{-1}(S(u(t))) \| \) may be made arbitrary small.

Using the same reasoning, it may be shown that \( |\dot{u} - \dot{u}^*| \) is arbitrary small, etc., which completes the proof.