Design of reliable and uncertain multirate control systems forced by bounded exogenous signals

Arturo Locatelli, Nicola Schiavoni
Dipartimento di Elettronica e Informazione
Politecnico di Milano
Piazza L. da Vinci, 32
20133 Milano, Italy
{locatell,schiavon}@elet.polimi.it

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Abstract

This paper considers the design of digital control systems, where the measurement mechanism is nonstandard, that is, each output is measured at its own rate. It addresses the classical regulator problem, under the assumptions that the exogenous signals are bounded, the instrumentation is subject to failures or the plant parameters undergo large variations. Some sufficient conditions are given for the existence of centralized and decentralized least-order regulators able to solve the problems outlined.

1 Introduction

Multirate control systems are characterized by the fact that each input is updated at its own rate and/or each output is measured at its own rate; as such, they are able to model many real life situations. These control systems have received a great deal of attention for many years [1].

This paper tackles the regulator problem, which entails asymptotic stability and zero error regulation, under additional robustness constraints. First, the case where some loops open because of failures in the instrumentation (sensors and actuators) is considered: the Reliable Regulator Problem (RRP) requires that stability and zero error regulation are maintained, to the maximum possible extent, even in these situations. Then, the design of a single regulator for a whole set of systems, representing various possible models of a partially uncertain plant, is considered in a second problem, called Simultaneous Regulator Problem (SRP). Constructive conditions are presented, sufficient to solve the above problems by means of least order regulators, under the assumption that the plant is asymptotically stable.

The considered updating mechanism is standard, since otherwise the problems do not admit any solution. Put in a different way, plants with infrequent output sampling are only considered.

The present results are also connected with those appearing in [4] and [5]. In particular, the here suggested regulators supply the corresponding control systems with a generalized form of the unconditional stability property defined in [4].

Next section presents the formal statement of RRP along with a constructive sufficient solvability condition. Analogously, Section 3 is entirely devoted to SRP. Section 4 extends the results of the preceding sections to the case of fully decentralized control structures. The paper ends with an example (Section 5) and some concluding remarks (Section 6). The proofs of all the results are collected in an Appendix.

2 Reliable regulator problem

The control system considered in this section is represented in Fig. 1. Loosely speaking, the problem consists of designing a stabilizing controller also ensuring zero error regulation, under nonvanishing bounded exogenous signals with a rational spectrum, both when the instrumentation is properly working and when a failure occurs. In order to properly face plant parameters variations, the controller must incorporate a reduplication of the model of the exogenous signals, which makes it not asymptotically stable.

![Figure 1: The multirate control system for the RRP.](image-url)
Turning to a formal description of the control system, the plant $P$ is asymptotically stable and given by

$$x_P(t+1) = A_P x_P(t) + B_P u(t) + E_P d(t)$$

$$y_P(t) = C_P x_P(t) + F_P d(t)$$

where $u_P \in R^m$, $x_P \in R^n$, $y_P \in R^m$, $d \in R^v$. Now assume that the $i$-th component of $y_P$, $i \in M := \{1,2,\ldots,m\}$, of the output vector $y_P$ is periodically measured at times $t = kT_{Ni} + \tau_{Ni}$, $k$ integer, $T_{Ni}$ positive and finite integer, $\tau_{Ni} < T_{Ni}$ nonnegative integer. This measurement mechanism can be modelled as the following system $N$:

$$y(t) = N(t) y_P(t)$$

(1.a)

$$N(t) := \text{diag}\{n_i(t)\} , n_i(t) = \begin{cases} 1 & t = kT_{Ni} + \tau_{Ni} \\ 0 & t \neq kT_{Ni} + \tau_{Ni} \end{cases}$$

(1.b)

Observe that the output $y_i$ is set to zero in the whole interval between two successive measurements. The sensor $S$ and actuator $A$ are described by

$$\tilde{y}(t) = d_{\text{sym}}[\sigma_1, \sigma_2, \ldots, \sigma_m] y(t)$$

$$u_P(t) = d_{\text{sym}}[\alpha_1, \alpha_2, \ldots, \alpha_m] u_P(t)$$

respectively, where $\sigma_i = 1$ ($\alpha_i = 1$) if the $i$-th sensor $S_i$ (actuator $A_i$) is properly working and $\sigma_i = 0$ ($\alpha_i = 0$) if a failure occurs. The reference signal $y_P^* \in R^m$ for $y_P$ and the disturbance $d$ are generated by the following system $E$, where

$$x_E(t+1) = A_E x_E(t)$$

(2.a)

$$y_P^*(t) = C_{EY} x_E(t)$$

(2.b)

$$d(t) = C_{ED} x_E(t)$$

(2.c)

and $x_E \in R^{nE}$. Some standing assumptions on $E$ are in order:

(i) Matrix $A_E$ is known, whereas vector $x_E(0)$ and matrices $C_{EY}$ and $C_{ED}$ are unknown;

(ii) The eigenvalues of $A_E$ are distinct and have magnitude equal to 1.

The regulator $R$ to be designed makes its output $\tilde{u}$ to depend on the sensed error

$$\tilde{e}(t) := y_o^*(t) - \tilde{y}(t)$$

where $y_o$ is the output of the system $N^0$ defined by

$$y_o^*(t) = N(t) y_P^*(t)$$

(3)

Observe that the $i$-th component of $\tilde{e}$ equals the $i$-th component of the measured error

$$e_i(t) := y_i^*(t) - y_i(t)$$

(4)

only when $S_i$ is properly working and the $i$-th component of $\tilde{u}$ equals the $i$-th component of $u_P$ only when $A_i$ is properly working. The regulator $R$ is basically linear, that is, the relationship it establishes between $\tilde{e}$ and $\tilde{u}$ is linear for each operating condition of the multivariable actuator and sensor. However, the regulator is assumed to know such a condition, that is, it knows the value of the $\alpha_i$'s and the $\sigma_i$'s, and, when a failure occurs in loop $l$, $l \in M$, it sets the effect of $\tilde{e}_i$ on $\tilde{u}$ to zero, which is equivalent to zeroing the $l$-th column of the transfer function of $R$. The rationale behind that is simple. For as concerns a failure of the $l$-th sensor, the variable $\tilde{e}_i$ is different from $e_i$ and does not contain any piece of information useful for control. As a consequence, letting a nonzero signal $\tilde{e}_i$ to actually influence the output of the regulator (which will include an appropriate reduplication of the model of the exogenous signals) would render impossible the zeroing of the error components different from the $l$-th one. With reference to a failure of the $l$-th actuator, observe that the $l$-th row of the regulator transfer function can be considered as set to zero in this case, without any effect on the behavior of the closed loop system. Then, note that no more than $m-1$ components or combinations of the outputs can be regulated by means of $m-1$ control variables. Therefore, it is advisable to renounce to control a well specified output in order to try to regulate all the remaining ones: the choice of the $l$-th component of $\tilde{e}$ is anyhow arbitrary. As a convention, for any $l$, $l \in M$, denote by $R[l]$ and $R[l||$ the minimal realizations of the transfer functions obtained by the one of $R$ after zeroing the $l$-th column and the $l$-th row and column, respectively.

The system viewed from the regulator is $T$-periodic, with

$$T := \min_{i \in M} \{T_{Ni}\}$$

(5)

Then, letting $M' := M - \{l\}$, $l \in M$, and the control system error $e_p \in R^m$ be

$$e_p(t) := y_P^*(t) - y_P(t)$$

(6)

the problem of this section is the following.

**Reliable regulator problem (RRP)**

Find a regulator $R$ such that:

(a) If no failures occur (i.e., $\alpha_i \sigma_i = 1$, $i \in M$)

(i) The closed loop system $(P,N,S,R,A)$ is asymptotically stable;

(ii) The regulation constraint

$$\lim_{t \to \infty} e_p(t) = 0$$

holds true for all the matrices $C_{EY}$, $C_{ED}$ and the vector $x_E(0)$, and the perturbations of matrices $A_P$, $B_P$, $C_P$, $E_P$ and $F_P$, which preserve the asymptotic stability of the closed loop system $(P,N,S,R,A)$;

(b) If a failure occurs in the $l$-th loop, $l \in M$ (i.e., $\alpha_i \sigma_i = 1$, $i \in M'$), $\alpha_i \sigma_i = 0$)

(i) The closed loop systems $(P,N,S,R_0|A)$ and $(P,N,S,R_1|A)$ are asymptotically stable;
(ii) The regulation constraint
\[
\lim_{t \to +\infty} e_P(t) = 0, \quad i \in M^I
\]
holds true for all the matrices \( C_{EY}, C_{ED} \) and the vector \( x_E(0) \), and the perturbations of matrices \( A_P, B_P, C_P, E_P \) and \( F_P \), which preserve the asymptotic stability of the closed loop systems \((P,N,S,R\|A)\) and \((P,N,S,R\|A)\).

It is worth noting that what is called for in (b) concerning stability is the maximum one can require; asymptotic stability of \((P,N,S,R,A)\) can never be achieved in failed conditions, since \( R \) will contain unreachable/unobservable parts which are not asymptotically stable.

Let \( \Theta \) be the set of the arguments of the eigenvalues of \( E \) with nonnegative imaginary part, i.e.,
\[
\Theta = \left\{ \vartheta_k \mid 0 \leq \vartheta_k \leq \pi, \det\left(e^{i\vartheta_k I} - A_E\right) = 0 \right\}
\]
with cardinality \( \nu_E \). Then, the following existence theorem can be proven, where \( G_{P\vartheta_k} := C_P\left(e^{i\vartheta_k I} - A_P\right)^{-1}B_P \), \( \vartheta_k \in \Theta \), and, for any matrix \( A, A(i,l) \) denotes the matrix obtained from \( A \) after deleting its \( i \)-th and \( l \)-th column; conventionally, if \( i = 0 \) (\( l = 0 \)) no row (column) is deleted.

**Theorem 1**

If there exist \( \nu_E \) complex \( m \times m \) matrices \( \Phi_k \) such that all the eigenvalues \( \lambda \) of the matrices \( G_{P\vartheta_k}(i,i)\Phi_k(i,i), G_{P\vartheta_k}(i,0)\Phi_k(0,i), \vartheta_k \in \Theta, \quad i \in [0] \cup M \), satisfy the condition
\[
\cos(\arg(\lambda) - \vartheta_k) > 0 \quad \vartheta_k \in \Theta
\]
then RRP admits a solution.

Observe that the nonsingularity of matrices \( G_{P\vartheta_k} \) is a necessary condition not only for the existence of \( \Phi_k \) (since otherwise \( \arg(\lambda) \) is not well defined for all \( \lambda \)), but also for the existence of a regulator satisfying part (a) of the statement of RRP.

Whenever the condition of Theorem 1 is verified, time invariant regulators \( R \) solving RRP can be identified up to the value of their gain, which must be sufficiently small, as stated in the following theorem, where
\[
N := \sum_{t=1}^N t
\]
is positive definite (see the definition of \( N(t) \)).

**Theorem 2**

For any set of matrices \( \Phi_k \) satisfying the condition of Theorem 1, there exists \( \varepsilon > 0 \) such that, for any \( \varepsilon \in (0,\varepsilon) \), a solution to RRP is provided by the regulator \( R \) with transfer function
\[
R(z) = \sum_{k=1}^\nu_E \frac{z^k}{z^2 - 2z\cos \vartheta_k + 1}\frac{(K_{k1} + K_{k2}z)}{N-1}
\]
where, for all \( k = 1,2,...,\nu_E \),
\[
K_{k1} := \frac{\Re[\Phi_k]\cos \vartheta_k + \Im[\Phi_k]\sin \vartheta_k}{\nu_E}
\]
\[
K_{k2} := \Re[\Phi_k]
\]
The regulators of Theorem 2 are of the least possible order, as they only contain the reduplicated model of the system generating the exogenous signals necessary to steer the control systems error to zero. Further, they supply the control systems with a generalized form of the unconditional stability property, which is defined in [4], in the sense that any diminution of \( \varepsilon \), with respect to a given value belonging to the interval \((0,\varepsilon)\), is still such that the RRP is solved.

![Figure 2: The multirate control system for the SRP.](image-url)

**3 Simultaneous regulator problem**

The control system considered in this section is represented in Fig. 2. Loosely speaking, the problem consists of designing a stabilizing controller also ensuring zero error regulation, under nonvanishing bounded exogenous signals with a rational spectrum, for a whole finite set of models of the plant, each one subject to perturbations. Again, the controller must incorporate a reduplication of the model of the exogenous signals, which makes it not asymptotically stable.

The plant under control is asymptotically stable and described by the \( s \) models \( P_h, \quad h \in S := [1,2,...,s] \),
\[
\begin{align*}
x^h_P(t+1) &= A^h_P x^h_P(t) + B^h_P u_P(t) + E^h_P d(t) \\
y_P(t) &= C^h_P x^h_P(t) + F^h_P d(t)
\end{align*}
\]
Further, the measurement mechanism is still described by the system \( N \) defined in eqns. (1) and again the reference signal \( y^o_P \in R^m \) for \( y_P \) and the disturbance \( d \) are as in eqn. (2). The regulator \( R \) for this problem is linear and
makes its output \( u_P \) to depend on the measured error \( e \) (eqn. 4). The system viewed from it is \( T \)-periodic (eqn. (5)). Finally, remembering the already given definitions of \( y^o \) and \( e_P \) (eqns. (3) and (6)), the problem of this section is the following.

**Simultaneous regulator problem (SRP)**

Find a single regulator \( R \) such that, for all \( h \in S \),

(i) The closed loop system \((P_{gl}N,R)\) is asymptotically stable;

(ii) The regulation constraint

\[
\lim_{t \to \infty} e_P(t) = 0
\]

holds true for all the matrices \( C_{EY}, C_{ED} \) and the vector \( x_E(0) \), and the perturbations of matrices \( A^hP, B^hP, C^hP, E^hP \) and \( F^hP \), which preserve the asymptotic stability of the closed loop system \((P_{gl}N,R)\).

Proceeding as in Section 2, for any \( \varrho_k \in \Theta \), let

\[
G^h_{P\varrho_k} := C^h_{P\varrho_k} e^{j\varrho_k I - A^h_{P\varrho_k}} B^h_{P\varrho_k} = \{ h_{P\varrho_k} \}, \quad h \in S. \quad \text{Then, the following results hold, where } N \text{ is as in eqn. (8).}
\]

**Theorem 3**

If there exist \( r_E \) complex \( m \times m \) matrices \( \Phi_k \) such that all the eigenvalues \( \lambda \) of the matrices \( G^h_{P\varrho_k} \Phi_k, h \in S, \varrho_k \in \Theta \), satisfy condition (7), then SRP admits a solution.

**Theorem 4**

For any set of matrices \( \Phi_k \) satisfying the condition of Theorem 3, there exists \( \varepsilon > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon) \), a solution to SRP is provided by the regulator \( R \) with transfer function (9), (10).

The regulators of Theorem 4 are of the least possible order, as they only contain the reduplicated model of the system generating the exogenous signals necessary to steer the control systems error to zero. Further, they supply the control systems with a generalized form of the unconditional stability property, which is defined in [4], in the sense that any diminution of \( e \), with respect to a given value belonging to the interval \((0, \varepsilon)\), is still such that the SRP is solved.

4 Decentralized control structures

The problems dealt with in the preceding sections can be extended in a fairly easy way to encompass the case of decentralized regulators, namely regulators with diagonal transfer function. Indeed, this situation is somehow simpler to treat, because an actuator failure has the effect of rendering impossible the zeroing of the corresponding error variables, but has no effect on the other errors, provided that stability is preserved. Therefore, there is no need to detect the status of the actuators and it can be proven that each local regulator must know only the status of the corresponding sensor. On the contrary, in a centralized control structure, there is a coupling of all regulator inputs and outputs.

Apart from the above considerations, the results for completely decentralized control structures are obtained from Theorems 1-4 by just adding the conditions that matrices \( \Phi_k \) are diagonal. Indeed, this fact implies that the regulator transfer function \( R(z) \) is diagonal (see eqns. (9), (10)). Moreover, sufficient solvability conditions for RRP and SRP can be stated in terms of the problems data under certain circumstances, which allow one to give explicit expressions for the \( \Phi_k \)'s.

In the forthcoming theorem \( \gamma \varrho_{ki} = \arg\{g_{P\varrho_{ki}}\} \), \( \varrho_k \in \Theta, \quad i \in M, \quad \gamma^h_{\varrho_{ki}} = \arg\{g^h_{P\varrho_{ki}}\}, \quad h \in S, \quad \varrho_k \in \Theta, \quad i \in M \); further, when \( G^h_{P\varrho_k} \in S, \varrho_k \in \Theta \), is diagonally dominant, for \( i \in M \) let

\[
\varrho^h_{\varrho_{ki}} = \begin{cases} \arccos\left( \sum_{l=i} |g^h_{p\varrho_{kl}}| / |g^h_{P\varrho_{ki}}| \right) & \text{col. dominance} \\ \arccos\left( \sum_{l=i} |g^h_{p\varrho_{kl}}| / |g^h_{P\varrho_{ki}}| \right) & \text{row dominance} \end{cases}
\]

**Theorem 5**

If matrices \( G^h_{P\varrho_k} \), \( \varrho_k \in \Theta \), are triangular and nonsingular or diagonally dominant, then the RRP admits a decentralized solution. Further, there exists \( \varepsilon > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon) \), a solution is provided by the regulator \( R \) with transfer function (9), (10) and

\[
\Phi_k = \text{diag}\left\{ e^{j\varrho_{ki}} \right\}, \quad \varrho_k \in \Theta
\]

**Theorem 6**

If matrices \( G^h_{P\varrho_k} \), \( h \in S, \varrho_k \in \Theta \), are triangular and nonsingular and there exist \( \gamma_{\varrho_{kim}} \) and \( \gamma_{\varrho_{kim}} \),

\[
\gamma_{\varrho_{kiM}} - \gamma_{\varrho_{kim}} < \pi, \quad \varrho_k \in \Theta, \quad i \in M
\]

such that

\[
e^{j\gamma_{\varrho_{ki}}} = a^h_{\varrho_{ki}} e^{j\gamma_{\varrho_{kim}}} + b^h_{\varrho_{ki}} e^{j\gamma_{\varrho_{kiM}}}
\]

\[
a^h_{\varrho_{ki}} \geq 0, \quad b^h_{\varrho_{ki}} \geq 0, \quad h \in S, \varrho_k \in \Theta, \quad i \in M
\]

then the SRP admits a decentralized solution. Further, there
exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, a solution is provided by the regulator $R$ with transfer function (9), (10) and

$$\Phi_k = \text{diag}\left\{ e^{j \langle \hat{y}_{\theta_k} - 2 \varepsilon \langle \hat{y}_{\theta_k} \rangle \rangle} \right\}, \quad \theta_k \in \Theta$$  \hspace{1cm} (11)

**Theorem 7**

If matrices $G_R \theta_k$, $h \in S$, $\theta_k \in \Theta$, are diagonally dominant and there exist $\gamma_{\theta_k}$ and $\gamma_{\theta_k}$, where $\gamma_{\theta_k} \in \Theta$, $i \in M$ such that

$$e^{j \langle \hat{y}_{\theta_k} + \hat{\theta}_k \rangle} = a_{\theta_k} e^{j \gamma_{\theta_k}} + b_{\theta_k} e^{j \gamma_{\theta_k}}$$

$$a_{\theta_k} \geq 0, \quad b_{\theta_k} \geq 0, \quad h \in S, \quad \theta_k \in \Theta, \quad i \in M$$

then the SRP admits a decentralized solution. Further, there exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, a solution is provided by the regulator $R$ with transfer function (9), (10) and (11).

Observe that in Theorems 6 and 7 checking for the existence of $\gamma_{\theta_k}$ and $\gamma_{\theta_k}$ (which are not uniquely defined) simply amounts to verifying that a suitable straight line passing through the origin leaves all the vectors $e^{j \gamma_{\theta_k} h}$, $h \in S$, in the same open half plane.

5 **Illustrative example**

Just to give a rough idea of how the above results can be applied, an example of SRP is now briefly discussed.

Consider the model of the binary distillation column presented in [10] and assume that only the dynamical matrix is uncertain ($s = 3$), so that

$$A_p = \begin{bmatrix} .8773 & -16288 \\ .0962 & 8413 \end{bmatrix}, \quad A_p = 1.05 A_p, \quad A_p = .95 A_p$$

$$B_p = \begin{bmatrix} .02602 & .00516 \\ .0735 & 1.348 \end{bmatrix}, \quad C_p - I_2, \quad h = 1, 2, 3$$

The output components can be measured alternately every 2 time instants, so that $n_1(0) = n_2(1) = 0$ and $n_1(1) = n_2(0) = 1$. Furthermore, let the exogenous signals be sums of steps and sinusoids with unitary angular frequency, i.e.,

$$\Theta = \{ \theta_1, \theta_2 \}, \quad \theta_1 = 0, \quad \theta_2 = 1$$

It can easily be checked that the condition of Theorem 3 is fulfilled by

$$\Phi_1 = \left( G_{1 \theta_1}^1 \right)^{-1}, \quad \Phi_2 = \left( G_{1 \theta_2}^1 \right)^{-1}$$

Then, a centralized regulator which solves SRP is designed according to Theorem 4 with $\varepsilon = .15$.

Finally, the resulting responses of the errors variables to the reference signal

$$y^2(t) = [2 - \sin(t) + 3\sin(t)]$$

are shown in Figs. 3 and 4, after a suitable decimation. Stability and zero error regulation are apparently attained, but the transients are very poorly damped.

![Figure 3: Responses of the first component of $e_p$.](image-url)
the problems formulation. Indeed, a typical design technique (particularly useful when decentralization and other constraints on the regulator structure are present) is to choose a suitably parametrized class of regulators and then seek the element which is the best one in some suitable sense. The actual optimization is usually performed by means of iterative procedures, to be initialized by regulators guaranteeing stability, regulation and robustness. Then, the results of the previous sections can be exploited whenever the adopted class of regulators includes those specified by eqns. (9), (10). Usually, some more dynamics are added to the simple reduplication of the internal model of the exogenous signals, which does not ensure satisfactory transients responses.

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References


6 Concluding remarks

With reference to multirate control systems, the paper considered some problems concerning various robustness issues. The theory can be easily extended to encompass some other interested cases, like the ones of multiple failures or of combined failures and uncertainties in the plant description. Further, in the special case where only the sensors are subject to failures, a necessary and sufficient solvability condition can be stated.

All the proposed solutions consist of least order small gain regulators, which are of interest even though they supply the control systems with sluggish transients, because the dynamical behavior has not been taken into account in


Appendix: Proofs of the theorems

Proof of Theorems 1 and 2 (a short account)

The proof actually consists of only showing the correctness of Theorem 2, as, then, Theorem 1 directly follows.

Consider first the Meyer-Burres time invariant reformation [6] of the $T$-periodic system resulting from the interconnection of $P, N, S, R, A, E$ and $NP$. In view of the Internal Model Principle [7], [8], the regulator specified in Theorem 2 guarantees the fulfillment of the robust regulation constraints (a)-(ii) and (b)-(ii) of the statement of RRP, for all the exogenous signals generated by $E$ whenever the time-invariant reformation $C$, $C_0|I$ and $C_I$ of the closed loop systems $(P, N, S, R, A)$, $(P, N, S, R|I, A)$ and $(P, N, S, R|I, A)$, $l \in M$, are asymptotically stable.

In order to prove the stability of $C$, observe that for $\varepsilon = 0$ its eigenvalues are those of the reformulated regulator, all on the boundary of the unit circle, and those of the reformulated plant, whose dynamical matrix is $A_p^T$. The standing stability assumption on $P$ entails that the eigenvalues of $A_p^T$ have magnitude less than 1 as well, so that the result follows if the branches of the "generalized multivariable root loci" starting from the poles of the reformulated regulator move towards the interior of the unit circle for small positive $\varepsilon$. By resorting to a power series expansion of the closed loop poles around $\varepsilon = 0$ it can be shown that this fact actually happens if conditions (7) is satisfied. On this regard, see the proof of Theorems 1 and 2 in [9].

Since the stability of $C_0|I$ and $C_I$ can be proven in a similar way, the proof is complete.

Proof of Theorems 3 and 4 (a short account)

The proof is similar to that of Theorems 1 and 2. This time, reference is made to the time invariant reformation of the $T$-periodic systems resulting from the interconnection of $P, N, R, E$ and $NP$. Then, again in view of the Internal Model Principle [7], [8], the regulation constraint (ii) of the statement of SRP issubstituted to the asymptotic stability of the reformation of the closed loop systems $(P, N, R)$, $h \in S$, which can be proven by following the same line of reasoning as above.

Proof of Theorem 5

When the matrices $G_p \gamma_k$, $\gamma_k \in \Theta$, are triangular and nonsingular, the eigenvalues of the matrices $G_p \gamma_k(i,i) \gamma_k(i, i)$ and $G_p \gamma_k(i,0) \gamma_k(0, i)$, $\gamma_k \in \Theta$, $i \in [0] \cup M$, take on the form $g_p \gamma_k h e^{ij\bar{\gamma}_k}$, so that condition (7) is planly satisfied.

Assume now that the matrices $G_p \gamma_k$, $\gamma_k \in \Theta$, are column diagonally dominant. In view of the Gershgorin's Theorem, the eigenvalues of the matrices $G_p \gamma_k(i,i) \gamma_k(i, i)$ and $G_p \gamma_k(i,0) \gamma_k(0, i)$, $\gamma_k \in \Theta$, $i \in [0] \cup M$, lie in the region

$$\Omega_{\gamma_k} = \bigcup_{i \in M} \left( g_p \gamma_k h e^{ij\bar{\gamma}_k} + \gamma_k \sum_{k \in \mathbb{N}} \left| g_p \gamma_k h e^{ij\bar{\gamma}_k} \right|^2 \right)
0 \leq \rho_1 \leq 1, 0 \leq \xi < 2\pi$$

Cumberson computations show that, for any $\omega \gamma_k \in \Omega_{\gamma_k}$,

$$\cos(\arg(\omega \gamma_k) - \gamma_k) = \cos \left( \arg \left( \sum_{k \in \mathbb{N}} \left| g_p \gamma_k h e^{ij\bar{\gamma}_k} \right|^2 \right) \right)$$

for some $i$, $i \in M$, $\rho_1$, $0 \leq \rho_1 \leq 1$ and $\gamma$, $0 \leq \gamma < 2\pi$, so that condition (7) holds.

Since the case of row diagonal dominance can be treated in a completely similar way, the proof follows in view of Theorems 3 and 4.

Proof of Theorem 6

The eigenvalues of the matrices $G_p \gamma_k$, $h \in S$, $\gamma_k \in \Theta$, take on the form $g_p \gamma_k h e^{ij\bar{\gamma}_k}$, $0 \leq \rho_1 \leq 1$ and $\gamma$, $0 \leq \gamma < 2\pi$, so that condition (7) is satisfied. Then, the proof follows in view of Theorems 3 and 4.

Proof of Theorem 7

Assume that the matrices $G_p \gamma_k$, $h \in S$, $\gamma_k \in \Theta$, are column diagonally dominant. In view of the Gershgorin's Theorem, the eigenvalues of the matrices $G_p \gamma_k h \in S$, $\gamma_k \in \Theta$, lie in the regions

$$\Omega_{\gamma_k} = \bigcup_{i \in M} \left( g_p \gamma_k h e^{ij\bar{\gamma}_k} + \gamma_k \sum_{k \in \mathbb{N}} \left| g_p \gamma_k h e^{ij\bar{\gamma}_k} \right|^2 \right)
0 \leq \rho_1 \leq 1, 0 \leq \xi < 2\pi$$

Cumberson computations show that, for any $\omega \gamma_k \in \Omega_{\gamma_k}$,
\[
\cos(\arg(\xi^h_{g_k}) - \varphi_k) = \cos\left(\gamma^h_{g_{ki}} - \frac{5(\gamma_{k,im} + \gamma_{k,im})}{\sum_{l=1}^{M} \left| \xi^h_{p,g_{kl}} \right|} + \arg \left(1 + \rho_l \frac{\sum_{l=1}^{M} \xi^h_{p,g_{kl}}}{\sum_{l=1}^{M} \xi^h_{p,g_{kl}}} e^{j\zeta} \right)\right)
\]
for some \(i, i \in M, \rho_l, 0 \leq \rho_l \leq 1\) and \(\zeta, 0 \leq \zeta < 2\pi\).

Further, in view of the checking of condition (7), it can be shown that the worst cases with respect to \(\zeta\) arise when
\[
\arg \left(1 + \rho_l \frac{\sum_{l=1}^{M} \xi^h_{p,g_{kl}}}{\sum_{l=1}^{M} \xi^h_{p,g_{kl}}} e^{j\zeta} \right) = \pm \varphi^h_{g_{ki}}
\]

In correspondence to \(\pm \varphi^h_{g_{ki}}\) it turns out that
\[
\cos(\arg(\xi^h_{g_k}) - \varphi_k) = \begin{cases} 
\cos\left(5(\gamma_{k,im} - \gamma_{k,im})\right) \\
\cos\left(5(\gamma_{k,im} - \gamma_{k,im})\right)
\end{cases}
\]
so that condition (7) holds.

Since the case of row diagonal dominance can be treated in a completely similar way, the proof follows in view of Theorems 3 and 4.