The Hamiltonian structure of a 2-D rigid cylinder interacting
dynamically with \( N \) point vortices

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Abstract

This paper gives the equations of motion of the system of a 2-D rigid cylinder of general cross-sectional shape interacting dynamically with \( N \) point vortices and shows that for circular shapes, the equations are Hamiltonian with respect to a Poisson bracket structure that is the sum of the Lie-Poisson bracket on \( \mathfrak{se}(2)^* \), the dual of the Lie algebra of the Euclidean group on the plane, and the canonical Poisson bracket of \( N \) point vortices in an unbounded plane. We use this Hamiltonian structure to study some stability and control issues.

1 Introduction

We investigate the Hamiltonian structure of a 2-D rigid cylinder which interacts dynamically with \( N \) point vortices external to it. The problem is part of a larger project to understand the geometry, dynamics and control of a 3-D, deformable body moving in the vortical field of an incompressible fluid. Our motivation comes from several areas in engineering and physics where such investigations have relevance and applications, such as: the design of remotely piloted underwater vehicles [1], the locomotion of fish that shed vortices by flapping their tails [2], the dynamics of bubbles [3] and the interaction of fixed wing aircraft with their trailing vortex filaments. Theoretical investigations in these areas are not new. There are several papers that calculate forces and moments on moving bodies in vortical fields, for example [4, 5], but these do not consider the dynamics of the interacting fluid-solid system. The work of Galper and Miloh [6, 7] has a dynamics perspective, however, they extend Kirchhoff’s equations of motion to the case of a non-uniform potential flow field superimposed on the potential field associated with the moving rigid or deformable body. Extension to vortical fields is not considered. Our viewpoint, which differs from all these, is that of geometric mechanics and our focus is on the role of vorticity. Interacting fluid-solid systems in such a framework have not been well-studied. Indeed, the authors are not aware of the existence in literature of even the equations of motion of the system we are considering. The work that comes closest to addressing our problem is Koiller [8]. We hope to subsequently apply to these problems the many ideas in nonlinear stability, relative equilibria and control that have been developed in the general geometric theory of mechanics [9].

The system of a 2-D rigid cylinder interacting dynamically with \( N \) point vortices may be viewed, in the context of geometric mechanics, as the blend of two simpler, classical systems each with a well-known Hamiltonian structure. One is the system of a 2-D rigid cylinder moving in a field with zero vorticity. The equations of motion of this system, derived by Kirchhoff, can be shown to be Hamiltonian [1] with respect to the Lie-Poisson bracket structure on \( \mathfrak{se}(2)^* \), the dual of the Lie algebra of the Euclidean group on the plane. The other system is that of \( N \) point vortices moving externally to a closed, rigid, stationary boundary. The equations of the vortices were shown by C. C. Lin [10] to be Hamiltonian with respect to the same canonical symplectic structure as that of \( N \) vortices in an unbounded plane. We present in this paper the equations of motion of the dynamically interacting system for a cylinder of general cross-sectional shape and show that, at least for circular shapes, the equations are Hamiltonian with respect to a Poisson bracket structure that is simply the sum of the brackets of the two, simpler systems referred to above i.e. Lie-Poisson plus canonical point vortex. The equations of motion are derived from a standard momentum balance analysis in the plane. The flow is assumed to be inviscid, incompressible, at rest at infinity and satisfies the zero normal velocity condition on the body. In the last subsection of this paper the Hamiltonian structure is used to identify some stability and control issues, in particular those related to the classical Föppl equilibrium [11, 12]. These will be studied in more detail in the future.
2 The equations of motion of a 2-D rigid cylinder in an inviscid, incompressible, vortical field

2.1 Smooth vorticity fields

The equations are derived for a smooth vorticity field and then specialized to a field of point vortices. A schematic sketch in the case of point vortices is shown in Fig.1.

Figure 1: A 2-D rigid cylinder interacting dynamically with point vortices. XY is a reference frame fixed in space, xy is a body-fixed frame with origin at the center of mass CM and axes parallel to the principal directions.

(a) Linear momentum

We start by deriving an expression for the linear momentum of the fluid. We make use of the following vector identity ([13],p.65):

\[ \int_{A} \mathbf{a} \, dA = \int \left( \mathbf{r} \times \text{curl} \mathbf{a} \right) \, dA + \oint_{\partial B} \mathbf{r} \times (\mathbf{n} \times \mathbf{a}) \, ds, \quad (1) \]

where \( \mathbf{a} \) is a divergence-free vector field on some bounded domain \( A \subset \mathbb{R}^2 \), \( \mathbf{r} \) is the position vector with respect to some fixed reference frame, \( \mathbf{n} \) is the unit inward normal vector on the boundary. Now let \( \mathbf{a} = \mathbf{u} \) = the velocity field of the flow. Let \( C_R \) denote a moving circular boundary of radius \( R \) centered on some arbitrary point in the body. \( C_R \) encloses the body and all of the vorticity (for all time). Let \( \partial B \) denote the moving boundary of the body. Then the momentum of the fluid (of constant, unit density) in the domain \( A_R \) between these two boundaries is:

\[ \int_{A_R} \mathbf{u} \, dA = \int_{A_R} (\omega \mathbf{r} \times \mathbf{k}) \, dA + \oint_{\partial B} \mathbf{r} \times (\mathbf{n} \times \mathbf{u}) \, ds, \]

\[ + \oint_{C_R} \mathbf{r} \times (\mathbf{n} \times \mathbf{u}) \, ds, \quad (2) \]

where \( \omega \mathbf{k} = \text{curl} \mathbf{u} \) is the vorticity field, \( \mathbf{k} \) being the unit vector normal to the plane. Note that the normal in the body contour integral points away from the body and the normal in the \( C_R \) contour integral points radially inward. Counter-clockwise circulation is considered positive with the associated vorticity vector pointing out of the plane.

Write

\[ \mathbf{u} = \nabla \Phi_B + \mathbf{u}_V, \quad (3) \]

where \( \nabla \Phi_B \) is the curl-free velocity field in \( \mathbb{R}^2 \setminus B (\mathbb{R}^2 \supset B = \text{region occupied by body}) \) determined uniquely by the motion of the body satisfying the boundary conditions:

\[ \nabla \Phi_B \cdot \mathbf{n} = q \cdot \mathbf{n} \text{ on } \partial B, \quad (4) \]

\[ \nabla \Phi_B \rightarrow 0, \quad R \rightarrow \infty, \quad (5) \]

where \( q \) is the velocity of the body boundary point. \( \mathbf{u}_V \) is the velocity field due to the vorticity satisfying the boundary conditions

\[ \mathbf{u}_V \cdot \mathbf{n} = 0 \text{ on } \partial B, \quad (6) \]

\[ \mathbf{u}_V \rightarrow 0, \quad R \rightarrow \infty. \quad (7) \]

It should be noted that \( \mathbf{u}_V = \mathbf{u}_0 + \mathbf{u}_I \). Here \( \mathbf{u}_0 \) is the velocity field due to the vorticity in the absence of boundaries and is naturally defined on all of \( \mathbb{R}^2 \) (\( \mathbf{u}_0 \rightarrow 0 \) as \( R \rightarrow \infty \), and \( \nabla \times \mathbf{u}_0 = 0 \) in \( B \)). \( \mathbf{u}_I \) is the velocity field that is curl-free in \( \mathbb{R}^2 \setminus B \) and is hence uniquely determined in \( \mathbb{R}^2 \setminus B \) by the boundary conditions:

\[ \mathbf{u}_I \cdot \mathbf{n} = -\mathbf{u}_0 \cdot \mathbf{n} \text{ on } \partial B, \quad (8) \]

\[ \mathbf{u}_I \rightarrow 0, \quad R \rightarrow \infty. \quad (9) \]

Now apply Newton’s second law to the fluid in \( A_R \). The following assumptions are made during the derivation: the force of gravity on the fluid is balanced by the hydrostatic pressure, there is no other external force on the fluid, the total vorticity in the fluid is constant in time, there is no circulation around the body and the weight of the body is balanced by the force of buoyancy. We further make the simplifying assumption that the fluid and body have constant, uniform density equal to unity. Hence,

\[ \mathbf{F}_S + \oint_{C_R} p_R \mathbf{n} ds = \frac{d}{dt} \int_{A_R} \mathbf{u} \, dA \]

\[ - \oint_{C_R} \mathbf{u}_I (\mathbf{u}_0 \cdot \mathbf{n}) \, ds, \quad (10) \]

where \( \mathbf{F}_S \) is the force (per unit span) exerted by the solid on the fluid at the boundary \( \partial B \) and is equal and opposite to that exerted by the fluid on the solid (denoted by \( -\mathbf{F}_S \)), \( \int_{C_R} p_R \mathbf{n} ds \) is the total contribution of the pressure forces acting on \( C_R \), and \( \oint_{C_R} \mathbf{u}_I (\mathbf{u}_0 \cdot \mathbf{n}) \, ds \) is the net flux of momentum across \( C_R \). Here \( \mathbf{u}_I = \mathbf{u} - \mathbf{U} \), where \( \mathbf{U} \) is the velocity of the body center of mass which is coincident with the center of the circle \( C_R \). Since \( -\mathbf{F}_S = A_R (d\mathbf{U}/dt) \), where \( A_R \) is the cross-sectional area of the cylinder, we get the following vector equation for the system comprising of a rigid body
and an incompressible, inviscid fluid in the domain $A_R$:

$$A_b \frac{dU}{dt} + \frac{d}{dt} \int_{\partial B} r \times (n \times \nabla \Phi_B) \, ds$$

$$+ \frac{d}{dt} \int_{A_R} (\omega r \times k) \, dA + \frac{d}{dt} \int_{\partial B} r \times (n \times u_V) \, ds$$

$$+ \mathbf{P}_R = 0, \quad (11)$$

where

$$\mathbf{P}_R = \frac{d}{dt} \int_{C_R} r \times (n \times u) \, dA - \int_{C_R} \mathbf{u} \cdot (\nabla \times \mathbf{u}) \, ds$$

$$- \int_{C_R} \oint_{\partial B} p_R \, n \, ds. \quad (12)$$

(b) Angular Momentum

We use the vector identity ([13], p.55, also see comment above eqn.(18) on p.65):

$$\int r \times a \, dA = -\frac{1}{2} \int (r^2 \text{curl} a) \, dA$$

$$- \frac{1}{2} \int r^2 (n \times a) \, ds, \quad (13)$$

where $r = ||r||$. Here again $n$ is the inward pointing unit normal. Hence the angular momentum of the fluid in the domain $A_R$ is:

$$\int_{A_R} r \times u \, dA = -\frac{1}{2} \int_{A_R} \omega r^2 \, dA$$

$$- \frac{1}{2} \int_{\partial B} r^2 (n \times u) \, ds - \frac{1}{2} \int_{C_R} r^2 (n \times u) \, ds. \quad (14)$$

Now apply Newton’s second law for angular momentum for the fluid in $A_R$:

$$\mathbf{M}_S + \oint_{C_R} p_R r \times n \, ds =$$

$$\frac{d}{dt} \int_{A_R} r \times u \, dA - \int_{C_R} r \times u \cdot (\nabla \times \mathbf{u}) \, ds, \quad (15)$$

where $\mathbf{M}_S$ is the torque exerted by the solid on the fluid and is equal and opposite to that exerted by the fluid on the solid. The other terms are analogous to those in the force equation. Since $-\mathbf{M}_S = d(A_b \mathbf{b} \times \mathbf{U} + I\Omega)/dt$, where $\mathbf{b}(t)$ is the position vector in the inertial frame of the center of mass of the body, we thus get the following scalar equation from the conservation of angular momentum for the system comprising of a rigid body and an incompressible, inviscid fluid in the domain $A_R$.

$$\frac{d}{dt} \left(A_b \left(\mathbf{b} \times \mathbf{U} + I\Omega\right)\right) - \frac{1}{2} \frac{d}{dt} \int_{\partial B} r^2 (n \times \nabla \Phi_B) \, ds$$

$$- \frac{1}{2} \frac{d}{dt} \int_{A_R} \omega \Omega^2 k \, dA - \frac{1}{2} \frac{d}{dt} \int_{\partial B} r^2 (n \times u_V) \, ds$$

$$+ \mathbf{M}_R = 0. \quad (16)$$

Here $I$ is the principal moment of inertia tensor and $\Omega = \Omega k$ is the angular velocity of the body (which can be identified as a scalar in this 2-D case). The first two terms represent the total angular momentum of the body with respect to the origin of the fixed reference frame, and

$$\mathbf{M}_R = -\frac{1}{2} \frac{d}{dt} \oint_{C_R} r^2 (n \times u) \, ds - \oint_{C_R} p_R (r \times n) \, ds$$

$$- \oint_{C_R} r \times u (\nabla \times \mathbf{u}) \, ds. \quad (17)$$

2.1.1 The contribution of the $\mathbf{P}_R$ and $\mathbf{M}_R$ terms

It can be shown that the terms $\mathbf{P}_R$ and $\mathbf{M}_R$ can be simplified to the following:

$$\mathbf{P}_R = -\frac{d}{dt} \left( \mathbf{b} \times \mathbf{k} \int_{A_R} \omega \, dA + O \left( \frac{1}{R} \right) \right) \quad (18)$$

$$\mathbf{M}_R = -\frac{1}{2} \frac{d}{dt} \left( \mathbf{b}, \mathbf{b} \right) \mathbf{k} \int_{A_R} \omega \, dA + O \left( \frac{1}{R^2} \right). \quad (19)$$

The details of this simplification are given below, the uninterested reader may directly skip to the next subsection.

The far field behaviour of the velocity field is given by:

$$\mathbf{u} = \left( \frac{\int_{A_R} \omega \, dA}{2\pi R} \right) \mathbf{s} + O \left( \frac{1}{R^2} \right), \quad (20)$$

$$= \mathbf{u}_s \mathbf{s} + \mathbf{u}^{(2)}, \quad (21)$$

where $\mathbf{s}$ is the unit tangent vector on $C_R$, and $r = \mathbf{b} - R\mathbf{n}$, $\mathbf{b}(t) = O(1)$ for all $t$. Here we have made use of the assumption that there is no net circulation about the body and hence $\int_B \omega \, dA = 0$. It follows from the above decomposition (21) that

$$\oint_{C_R} \mathbf{u}^{(2)} \cdot ds = -\mathbf{k} \cdot \oint_{C_R} \mathbf{n} \times \mathbf{u}^{(2)} \, ds = 0. \quad (22)$$

Using $\mathbf{s} \times \mathbf{n} = \mathbf{k}$, the integral in the first term in $\mathbf{P}_R$ is now evaluated:

$$\oint_{C_R} r \times (n \times u) \, ds = -\mathbf{b} \cdot \mathbf{k} \int_{A_R} \omega \, dA$$

$$- \oint_{C_R} R\mathbf{n} \times \left( n \times \mathbf{u}^{(2)} \right) \, ds + O \left( \frac{1}{R^2} \right). \quad (23)$$

In the irrotational region traversed by $C_R$, (21) can also be written as:

$$\mathbf{u} = \nabla \Phi = \nabla \Phi_V + \nabla \Phi^{(2)},$$

where $\Phi_V$ is the multiple-valued velocity potential, invariant in time, due to a single vortex of strength $\int_{A_R} \omega \, dA$, and where $\Phi^{(2)}$ is single-valued because of (22). Keeping in mind that $R\mathbf{n}$ can be viewed as the position vector in any frame based at the center of $C_R$, we use the identity (1) and the divergence theorem to get:

$$\oint_{C_R} R\mathbf{n} \times \left( n \times \mathbf{u}^{(2)} \right) \, ds =$$

$$\oint_{C_R} R\mathbf{n} \times \left( n \times \nabla \Phi^{(2)} \right) \, ds = -\oint_{C_R} \Phi^{(2)} \, n \, ds. \quad (24)$$
The leading order term in the second integral in \( P_R \) is easily seen to be \( O(1/R) \). Using Bernoulli’s theorem in the irrotational region traversed by \( C_R \), the pressure integral in \( P_R \) is written as:

\[
\oint_{C_R} p_R n ds = \oint_{C_R} \left( \frac{\partial \Phi}{\partial t} - \frac{|\mathbf{u}|^2}{2} + f(t) \right) n ds.
\]

Note that the motion of the \( C_R \) boundary does not affect this integral. The only \( O(1) \) contribution to the pressure integral comes from the first term on the right. It follows that

\[
\frac{d}{dt} \oint_{C_R} R \mathbf{n} \times (\mathbf{n} \times \mathbf{u}^{(2)}) \, ds + \oint_{C_R} \frac{\partial \Phi}{\partial t} \mathbf{n} ds = 0.
\]

(25)

and one obtains (18).

The evaluation of \( M_R \) proceeds on similar lines. Rewrite the first integral in \( M_R \) as:

\[
\frac{1}{2} \oint_{C_R} (\mathbf{r}, \mathbf{r}) \cdot (\mathbf{n} \times \mathbf{u}) \, ds = \mathbf{b} \times \oint_{C_R} R \mathbf{n} \times (\mathbf{n} \times \mathbf{u}^{(2)}) \, ds - \frac{1}{2} \oint_{C_R} (\mathbf{b}, \mathbf{b}) \, k \oint_{A_R} \omega dA - \frac{1}{2} R^2 \oint_{A_R} \omega dA.
\]

(26)

Under the given assumptions the last term is invariant in time. In the above we have made repeated use of the vector identity:

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C},
\]

(27)

and the result (22). The next term in \( M_R \) gives:

\[
\oint_{C_R} \mathbf{r} \times \mathbf{u} \cdot (\mathbf{u} \cdot \mathbf{n}) \, ds - \oint_{C_R} \mathbf{r} \times \mathbf{u} \cdot (\mathbf{U} \cdot \mathbf{n}) \, ds = \oint_{C_R} R \mathbf{n} \times \mathbf{u}^{(2)} \cdot (\mathbf{U} \cdot \mathbf{n}) \, ds + O \left( \frac{1}{R} \right).
\]

(28)

Thus we get

\[
M_R = -\mathbf{b} \times \left[ \frac{d}{dt} \oint_{C_R} R \mathbf{n} \times (\mathbf{n} \times \mathbf{u}^{(2)}) \, ds + \oint_{C_R} p_R n ds \right] - \mathbf{U} \times \oint_{C_R} R \mathbf{n} \times (\mathbf{n} \times \mathbf{u}^{(2)}) \, ds - \oint_{C_R} R \mathbf{n} \times \mathbf{u}^{(2)} \cdot (\mathbf{U} \cdot \mathbf{n}) \, ds + \frac{1}{2} \frac{d}{dt} \left( \mathbf{b} \cdot \mathbf{b} \right) \mathbf{k} \int_{A_R} \omega dA + O \left( \frac{1}{R} \right).
\]

(29)

The term in the rectangular parantheses is zero because of (25), the next two terms sum to zero as can be seen using the above vector identity, and one obtains (19).

### 2.2 Point vortices

Now assume that the given vorticity field is a singular distribution of \( N \) point vortices, as shown in Fig.1. The vector identities (1) and (13) are not directly applicable to the given fluid domain but to a modified domain in which one removes small circles centered around each point vortex. It can then be shown that the same vector identities hold with the vorticity written as a delta distribution, \( \omega(r_k) = \sum \Gamma_k \delta(r - r_k) \).

Substituting (18) and (19) into (11) and (16), the following equations in the limit \( R \to \infty \) are then obtained:

\[
A_b \frac{d}{dt} \oint_{\partial B} \mathbf{r} \times (\mathbf{n} \times \nabla \Phi_B) \, ds + \frac{d}{dt} \oint_{\partial B} \sum \Gamma_k \mathbf{r}_k \times \mathbf{k} - \frac{d}{dt} \left( \mathbf{b} \times \mathbf{b} \sum \Gamma_k \right) \, ds = 0,
\]

(30)

and

\[
\frac{d}{dt} \left( A_b \mathbf{b} \times \mathbf{U} + I \mathbf{\Omega} \right) - \frac{1}{2} \frac{d}{dt} \oint_{\partial B} r^2 (\mathbf{n} \times \nabla \Phi_B) \, ds - \frac{1}{2} \frac{d}{dt} \sum \Gamma_k r_k^2 \mathbf{k} + \frac{1}{2} \frac{d}{dt} \left( \mathbf{b} \times \mathbf{b} \sum \Gamma_k r_k^2 \mathbf{k} \right) \, ds - \frac{1}{2} \frac{d}{dt} \oint_{\partial B} r^2 (\mathbf{n} \times \mathbf{u}_V) \, ds = 0.
\]

(31)

### 2.3 Body-fixed frame

We now attempt to transfer (30) and (31) which were derived in a spatially fixed or inertial frame, \( XY \) in Fig.1, to equations in a body-fixed frame. We choose a principal axis frame with origin at the body center of mass, shown as \( xy \) in Fig.1. For a given point in the domain the position vector \( \mathbf{r} \) in the inertial frame is related to the position vector \( \mathbf{l} \) in the body-fixed frame by:

\[
\mathbf{r} = \mathbf{R}(t) \mathbf{l} + \mathbf{b}(t),
\]

where \( \mathbf{R}(t) \in SO(2) \) gives the orientation of the body-fixed frame with respect to the inertial frame, and \( \mathbf{b}(t) \in \mathbb{R}^2 \) is the position vector of the origin of the body-fixed frame measured in the inertial frame. Putting \( \mathbf{b}(t) = 0 \) in the above gives the law for transforming vectors of the same norm. Time derivatives in the inertial frame are related to time derivatives in the body-fixed frame as follows:

\[
\frac{da}{dt} = \mathbf{R}(t) \frac{dv}{dt} + \mathbf{R}(t) \left( \mathbf{\Omega} \times \mathbf{v} \right),
\]

where \( \mathbf{a} = \mathbf{R}(t) \mathbf{v} \) and \( \mathbf{\Omega} \) is the angular velocity of the body referred to the body-fixed frame. The following relation is used often:

\[
\dot{\mathbf{R}}(t) \mathbf{v} = \mathbf{R}(t) \left( \mathbf{\Omega} \times \mathbf{v} \right).
\]

Transferring (30) and (31) term by term using the above relations one finally obtains the linear and angular momentum equations as:
\[
\frac{d}{dt} + \Omega \times \mathbf{L} = 0, \\
\frac{dA}{dt} + \mathbf{V} \times \mathbf{L} = 0,
\]
where

\[
\mathbf{L} = A_b \mathbf{V} + \int_{\partial B} 1 \times (\mathbf{n}_b \times \nabla \Phi_B) \, ds \\
+ \sum \Gamma_k \mathbf{k} \\
+ \frac{1}{2} \int_{\partial B} l^2 (\mathbf{n}_b \times (\mathbf{u}_V)_b) \, ds,
\]

\[
A = I \Omega - \frac{1}{2} \int_{\partial B} l^2 (\mathbf{n}_b \times \mathbf{V} \Phi_B) \, ds \\
- \frac{1}{2} \sum \Gamma_k (l_k \times \mathbf{l}_k) \mathbf{k} \\
- \frac{1}{2} \int_{\partial B} l^2 (\mathbf{n}_b \times (\mathbf{u}_V)_b) \, ds,
\]

\[
\mathbf{U} = R(t) \mathbf{V}
\]
and the subscript \( b \) denotes reference to the body-fixed frame. The expressions for \( \mathbf{L} \) and \( A \) can be written more elegantly as follows. Recall [14] that \( \Phi_B \) for \( f, g \) be shown that:

\[
\psi_B(x, y) = \frac{1}{2} \sum \Gamma_k^2 g(x_k, y_k; x_j, y_j),
\]

\[
W' = \sum \Gamma_k \psi_B(x_k, y_k) \\
+ \sum_{k,j(k>j)} \Gamma_k \Gamma_j G(x_k, y_k; x_j, y_j) \\
+ \frac{1}{2} \sum \Gamma_k^2 g(x_k, y_k; x_j, y_j),
\]

with \( G \) being a Green’s function satisfying appropriate boundary conditions and of the form

\[
G(x, y; x_0, y_0) = g(x, y; x_0, y_0) \\
+ (1/4\pi) \log[(x - x_0)^2 + (y - y_0)^2],
\]

and \( \psi_B \) is the stream function due to agencies other than the point vortices. The function \( g \) is harmonic everywhere in the fluid domain and is the stream function of the irrotational velocity field \( \mathbf{u}_f \), see (8), which annuls the non-zero normal velocities on the body due to the external vortices. All three functions \( G, g \) and \( \psi_B \) depend on the body shape. Lin [10] derived these equations for fixed boundaries. \( W \) is an invariant of the motion if \( \psi_B \) has no explicit time dependence. The theory remains valid for moving boundaries but \( W \) in general will no longer be an invariant. Denote it by:

\[
W'(\mathbf{r}_k, t) = \sum \Gamma_k \psi_B'(\mathbf{r}_k, t) \\
+ \sum_{k,j(k>j)} \Gamma_k \Gamma_j G'(\mathbf{r}_k; \mathbf{r}_j; t) + \frac{1}{2} \sum \Gamma_k^2 g'(\mathbf{r}_k; \mathbf{r}_k; t),
\]
where for any given t the functions $G'$ and $g'$ satisfy the same properties as $G$ and $g$. To write $W'$ in terms of $l_k, V(t)$ and $\Omega(t)$, note that the term $\psi_B'(r, t)$, which in this problem is solely due to the motion of the body, can be written in body-fixed coordinates as

$$
\psi_B'(r, t) = \psi_B(I_1, V(t), \Omega(t)),
$$

(48)

$$
= V(t) \cdot \eta(I) + \Omega(t) \cdot \kappa(I). 
$$

(49)

The fields $\eta(I)$ (of 2-vectors) and $\kappa(I)$ (of 1-vectors) depend only on the shape of the body. Their components are the harmonic conjugates of the Kirchhoff potentials that appear in the analogous linear decomposition of the potential function of the irrotational flow associated with the motion of the body (36).

We make the following claim for $G'$ and $g'$. Let $r$ denote the position vector in the fixed frame and $I$ the position vector in the body-fixed frame as before. Then

**Proposition:**

$$
G'(r; r_0; t) = G(I; l_0),
$$

(50)

$$
g'(r; r_0; t) := G'(r; r_0; t) - 1/(2\pi) \log ||r - r_0||,
$$

$$
= G(I; l_0) - 1/(2\pi) \log ||I - l_0||,
$$

(48)

$$
= g(I; l_0).
$$

(51)

**Proof:** Check that $G'$ satisfies all properties as outlined by Lin [10] for all t. Note that

$$
\nabla G' = R(t) \nabla_b G,
$$

$$
\nabla g' = R(t) \nabla_b g,
$$

$$
\nabla \psi_B'(r, t) = R(t) \nabla_b \psi_B'(R(t)I + b(t), t),
$$

$$
= R(t) \nabla_b \psi_B(I, t),
$$

$$
\nabla^2 g' = \nabla^2_b g.
$$

i) $\nabla^2 g' = \nabla^2_b g = 0$. Hence $g'(r; r_0; t)$ is harmonic in the domain.

ii) The condition of zero circulation around the body is:

$$
\oint_{\partial B} \frac{\partial G'}{\partial n} ds = \oint_{\partial B} \nabla G' \cdot n ds = 0.
$$

This is satisfied since

$$
\oint_{\partial B} \nabla G' \cdot n ds = \oint_{\partial B} R(t) \nabla_b G \cdot R(t) n_0 ds,
$$

$$
= \oint_{\partial B} \nabla_b G \cdot n_0 ds = 0.
$$

iii) The far-field behaviour of $G'$ should be:

$$
G'(r; r_0; t) = \frac{1}{2\pi} \log ||r - r_0|| + O\left(\frac{1}{||r - r_0||}\right),
$$

$$
\frac{\partial G'}{\partial s} = O\left(\frac{1}{||r - r_0||^2}\right),
$$

$$
\frac{\partial G'}{\partial n} = \frac{1}{2\pi ||r - r_0||} + O\left(\frac{1}{||r - r_0||^2}\right).
$$

Since $||r - r_0|| = ||1 - l_0||$, and using the relations between gradients and vectors in the two frames, one sees that $G'$ does possess the above behavior $\blacksquare$

Thus,

$$
W'(r, t) = W(I, V(t), \Omega(t))
$$

$$
= \sum \Gamma_k \psi_B(I_k, V(t), \Omega(t))
$$

$$
+ \sum \Gamma_k \Gamma_j G(I_k; I_j)
$$

$$
+ \frac{1}{2} \sum \Gamma_k^j g(I_k; I_j),
$$

(52)

and

$$
\frac{\partial W'}{\partial r} = R(t) \frac{\partial W}{\partial l_k}.
$$

(53)

The equations of motion of the vortices in the body-fixed frame can then be derived from (43) and (44) using the above results. For $k = 1, \ldots, N$, this gives:

$$
\Gamma_k R(t) \frac{d}{dt} I_k + \Omega \times I_k + V = J R(t) \frac{\partial W}{\partial l_k},
$$

$$
\Gamma_k \frac{d}{dt} I_k + \Omega \times I_k + V = J \frac{\partial W}{\partial l_k}.
$$

Here $J$ is the matrix:

$$
J = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
$$

Thus, to summarize, the equations of motion of the dynamically interacting system of a 2-D rigid cylinder and $N$ point vortices external to it are:

$$
\frac{d}{dt} I_k + \Omega \times I_k + V = J \frac{\partial W}{\partial l_k},
$$

(54)

$$
\frac{dA}{dt} + V \times L = 0,
$$

(55)

$$
\Gamma_k \left( \frac{d l_k}{dt} + \Omega \times I_k + V \right) = J \left( \frac{\partial W}{\partial l_k} \right),
$$

(56)

where $V$ is the velocity of the body center of mass referred to the body-fixed frame, $\Omega$ is the body rotational velocity, $L$ and $A$ are the momenta of the system given by (42) and $W$ is the Kirchhoff-Routh function generalized to moving boundaries and given by (52).

This is a $2N + 3$-dimensional system in the variables $L, A$ and $l_k(k = 1, \ldots, N)$. When the vortices are absent, the equations reduce to Kirchhoff’s equations of motion.

### 2.5 Hamiltonian form for the dynamics of a moving circular cylinder of radius $R$, and $N$ point vortices

The Hamiltonian structure of equations (54), (55) and (56) for general body shapes is, as yet, unknown. However, for
the case of a circular cylinder a fairly simple Hamiltonian structure emerges. The details are presented in this subsection.

Let the velocity of the cylinder center of mass be \( \mathbf{V}(t) = (u(t), v(t)) \). Then

\[
\sum \Gamma_k \psi_B(l_k, \mathbf{V}) = R^2 \sum \Gamma_k \left( \left( -\frac{\sin \theta_k}{l_k}, \frac{\cos \theta_k}{l_k} \right), \mathbf{V} \right),
\]

where \((l_k, \theta_k)\) are polar coordinates of the \(k\)th vortex in the body-fixed frame. Note that \(\psi_B\) is independent of \(\Omega\) since the rotation of the circular cylinder has no effect on the fluid. Conversely, the fluid too can have no effect on \(\Omega\) since the pressure forces act through the center of the cylinder. Therefore, the equations of motion should give \(d\Omega/d\tau = 0\) and this can indeed be confirmed.

The functions \(G\) and \(g\) for a circular cylinder can be calculated using the classical circle theorem of Milne-Thomson [14]. This gives a simple representation of the image vorticity in terms of two point vortices – one of the same strength but opposite sign at the inverse point and the other of the same strength and sign at the center of the circle – for each point vortex outside the circle. Thus,

\[
g(x, y; x_k, y_k) = \frac{1}{2\pi} \log |(x, y)| - \frac{1}{2\pi} \log |(x, y) - (R^2 x_k/l_k^2, R^2 y_k/l_k^2)|,
\]

where \(l_k^2 = x_k^2 + y_k^2\). Using (46) and (57) the function \(W\) can then be easily calculated. For future reference we write:

\[
W = \sum \Gamma_k \psi_B(l_k, \mathbf{V}) + W_G,
\]

where

\[
W_G = \sum_{k,j(k>j)} \Gamma_k \Gamma_j G(l_k; l_j) + \frac{1}{2} \sum \Gamma_k^2 g(l_k; l_k).
\]

Evaluating the mass matrix \(M\) shows that all off-diagonal terms vanish and, further, the first two diagonal terms are the same and are each equal to the mass plus added mass of the system. Denoting these terms by \(c\), \(M\) simplifies to

\[
M = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & I \end{bmatrix}
\]

where \(c = m + \pi R^2\). Therefore,

\[
\mathbf{L} = c\mathbf{V} + \mathbf{p}, \\
A = \pi,
\]

assuming \(\Omega(0) = 0\). Next, calculate:

\[
p = \sum \Gamma_k l_k \times \mathbf{k} + \oint_{\partial B} \mathbf{l} \times (\mathbf{n}_b \times (\mathbf{u}_V)_{b}) \, ds,
\]

\[
\pi = -\frac{1}{2} \sum \Gamma_k (l_k, l_k) \mathbf{k} - \frac{1}{2} \oint_{\partial B} l^2 (\mathbf{n}_b \times (\mathbf{u}_V)_{b}) \, ds.
\]

In this problem \(l = Rn_b\) on the body boundary. The contour integral in \(p\) simplifies as:

\[
\oint_{\partial B} \mathbf{l} \times (\mathbf{n}_b \times (\mathbf{u}_V)_{b}) \, ds = -R \oint_{\partial B} (\mathbf{u}_V)_{b} \, ds,
\]

and that in \(\pi\) vanishes:

\[
\oint_{\partial B} l^2 (\mathbf{n}_b \times (\mathbf{u}_V)_{b}) \, ds = 0.
\]

It can be checked after performing the necessary integrations that:

\[
\oint_{\partial B} (\mathbf{u}_V)_{b} \, ds = \sum \mathbf{k} \times \Gamma_k \left( -\frac{R}{l_k} \cos \theta_k, -\frac{R}{l_k} \sin \theta_k \right).
\]

Comparing with (57) it is seen that the following relation holds in this problem:

\[
\sum \Gamma_k \psi_B(l_k, \mathbf{V}) = \oint_{\partial B} \mathbf{l} \times (\mathbf{n}_b \times (\mathbf{u}_V)_{b}) \, ds.
\]

The general significance of this relation is not yet understood but it plays a simplifying role when constructing the Hamiltonian structure of this system. It is conjectured that an insight into this relation may help understand the Hamiltonian structure for general body shapes.

**Proposition:** The freely interacting system of a rigid circular cylinder of radius \(R\), and \(N\) point vortices external to it, in an incompressible, inviscid fluid is governed by the following system of equations:

\[
\frac{d\mathbf{L}}{dt} = 0,
\]

\[
\frac{dA}{dt} + \mathbf{V} \times \mathbf{L} = 0,
\]

\[
\Gamma_k \frac{dl_k}{dt} = -\oint_{\partial B} \frac{\partial H}{\partial l_k}, \quad k = 1, \ldots, N
\]

where

\[
\mathbf{L}(\tau) = \mathbf{L}(0) = c\mathbf{V} + \sum \Gamma_k l_k \times \mathbf{k}
\]

\[
+ R^2 \sum \mathbf{k} \times \Gamma_k \left( \frac{x_k}{x_k^2 + y_k^2}, \frac{y_k}{x_k^2 + y_k^2} \right),
\]
and

\[ H(L, l_k) = -W(L, l_k) + \frac{1}{2c} (L, L) \]

\[-\frac{1}{c} \left( \sum \Gamma_k (L \times l_k) \cdot k - \frac{1}{2} \sum \sum \Gamma_k^2 (l_k, l_k) \right) - \sum \sum \Gamma_k \Gamma_j (l_k, l_j) + \frac{R^4}{2} \left( \sum \Gamma_k (\frac{l_k}{l_k, l_k}) \right) \]. \quad (66)

\[ W(L, l_k) \] is the Kirchhoff-Routh function for the system and is given by (59) with \( V \) re-written in terms of \( L \) and \( l_k \).

This Poisson bracket on the state space \( \mathbb{R}^{2N} \setminus (\triangle \cup B) = P_0 \times P_u \) equipped with the following Poisson bracket. For \( F, G \in C^\infty(P) \),

\[ \{F, G\} = \{ F | p_\mu, G | p_\mu \}_{\text{Lie-Poisson}} + \{ F | p_\mu, G | p_\mu \}_{\text{point vortex}}. \]

Therefore if \( p(t) = (\mu(t), l_k(t)) \in P \) is an integral curve of the system, where \( \mu(t) = (L(t), A(t)) \), then

\[ \frac{dF}{dt} := \left\langle \nabla_p F, \frac{dp}{dt} \right\rangle = \left\langle \nabla_\mu F, \text{ad}^*_{\frac{\partial H}{\partial \mu}} \mu \right\rangle - \sum_{k=1}^N \left\langle \nabla_k F, J^{-1} \nabla_k (H/\Gamma_k) \right\rangle. \]

**Proof:** An exercise in verifying that the right vector field is the Lie-Poisson equations on \( \mathfrak{e}(2)^* \) are given by:

\[ \frac{d\mu}{dt} = \text{ad}_{\frac{\partial H}{\partial \mu}}^* \mu, \quad \mu \in \mathfrak{g}^*, \quad \delta \mu / \delta \mu \in \mathfrak{g} \]

for the Hamiltonian \( H \) and where:

\[ \text{ad}_{(\delta, \omega)}^* (\alpha, s) = (\alpha, J(\omega), -\delta s). \]

Making the identification \( \mu = (\alpha, s) = (L, A) \),

\[ \text{ad}_{\frac{\partial H}{\partial \alpha, \partial H / \partial L}}^* (A, L) = \left( \langle L, J \frac{\partial H}{\partial L} \rangle, -\frac{\partial H}{\partial A} J L \right). \]

Now if the momentum equations (63) and (64) are Lie-Poisson, we should have

\[ V \times L = \left\langle L, J \frac{\partial H}{\partial L} \right\rangle, \]

\[ 0 = \frac{\partial H}{\partial A} J L. \]

These relations are satisfied if:

\[ \frac{\partial H}{\partial A} = 0, \]

\[ \frac{\partial H}{\partial L} = V. \]

**Comments:**

1. \( \triangle \) in the definition of the phase space \( P \) is the set of collision points of vortices and \( B \) is the region occupied by the circle.

2. The system reduces to the correct Hamiltonian system in the two well-known cases: (i) the irrotational case i.e. no point vortices in the flow and (ii) the stationary body case. In case (i), one obtains the equations of motion for the body as \( dV / dt = 0 \) and \( H = (1/2c) (L, L) = (c/2) (V, V) \) Kirchhoff’s equations give exactly the same result. In case (ii) \( V = 0 \) and the terms within the large parantheses in (66) reduce to \( 1/2(L, L) \) and one obtains \( H = -W_G(l_k) \). The system thus reduces to the one investigated by C. C. Lin [10].

3. The Hamiltonian can be re-written in terms of \( V \) and \( I_k \) as:

\[ H(V, I_k) = -\sum \Gamma_k \psi_B(V, I_k) - W_G(I_k) \]

\[ + \frac{1}{2c} (cV + p, cV + p) \]

\[ -\frac{1}{c} \left( cV \times \left( \sum \Gamma_k I_k \right) \cdot k + \frac{1}{2} (p, p) \right) \]

\[ = -W_G(I_k) + \frac{c}{2} (V, V), \]

Using (3) and the \( L^2 \)-orthogonality of \( u_V \) and \( \nabla \psi_B \) it can be checked that the above is the total kinetic energy of the flow field minus the infinite contributions associated with the point vortex velocity field.

### 2.6 Stability and control

A brief introduction to the work currently in progress is presented in this section.

#### 2.6.1 Nonlinear stability of the Föppl equilibrium

Consider Föppl’s results [11, 12, 13] for equilibria of the system of a circular cylinder in an ambient uniform stream of velocity \( V \) and two counter-rotating point vortices of equal strength behind the cylinder located symmetrically with respect to the freestream direction. The same equilibrium holds if the cylinder moves with velocity \( V \) in a fluid at rest at infinity, the point vortices now move with the cylinder at the same velocity and are stationary in the body-fixed frame, as shown in Fig. 2.

The loci of equilibrium positions are described by the curves:

\[ l_0^2 - R^2 = \pm 2l_0 y_0, \quad (67) \]

where \( l_0^2 = x_0^2 + y_0^2 \), \((x_0, y_0)\) and \((x_0, -y_0)\) being the positions of the two vortices in the body-fixed frame. At each
equilibrium position, there is a linear relation between the vortex strength $\Gamma$ and $V$:

$$\Gamma = 4\pi V y_0 \frac{l_0^3 - R^4}{l_0^4}$$

(68)

It has been conjectured [2] that certain types of fish achieve locomotion by symmetrically shedding vortices in their wake. Stability and control of the above equilibrium may provide some insight into this phenomenon. The Hamiltonian structure described in the last section strongly suggests the use of the energy-Casimir method [9] to study the nonlinear stability (in the Lyapunov sense) of this equilibrium. This involves showing the existence of $\Phi(C)$, where $C$ is a Casimir function of the system, such that the first variation of the augmented Hamiltonian function

$$H_Q = H + \Phi(C)$$

(69)

vanishes at the Föppl equilibrium and the second variation quadratic form is positive or negative definite.

2.6.2 Control using external forces and moments on cylinder

It is obvious that, depending on the applications being considered, many different agendas for controlling the system of a cylinder and point vortices can be proposed. For example, in the problem of fish locomotion one could look at controlled smooth deformations of the cylinder or changes in body shape using controlled flaps. Analytically, however, the simplest non-trivial control problem to consider seems to be one where an external force and moment act through/about the center of mass of the cylinder.

It can be shown easily that in such a case the momentum equations (54) and (55) are:

$$\left( \frac{d}{dt} + \Omega \times \right) L = F_{ext},$$

(70)

$$\frac{dA}{dt} + V \times L = M_{ext}$$

(71)

with $L$ and $A$ defined as before. The point vortex equations (56) are unaltered.

For the case of a circular cylinder whose Hamiltonian structure was presented in the last section the above system of equations with an external control force $F_{ext}$ can be written as an affine Hamiltonian input-output system [15]. The moment control is absent since rotation of the cylinder is irrelevant. To this effect we write the control force $F_{ext} = (F_x, F_y)$ as:

$$F_x = L_y u(t),$$

$$F_y = -L_x u(t),$$

where $L = (L_x, L_y)$. Then it is easy to see that:

$$\frac{dL}{dt} = F_{ext},$$

$$\frac{dA}{dt} + V \times L = 0,$$

$$\Gamma_k \frac{dl_k}{dt} = -J \frac{\partial H}{\partial l_k}, \quad k = 1, \ldots, N$$

$$y = -A$$

with

$$H = H_0 + A u(t),$$

where $H_0$ is given by (66) is such an affine Hamiltonian input-output system, with $u(t)$ the input and $y$ the output. Note that the output depends only on the positions of the vortices. Here $(-L_y, L_x, 0, 0, \ldots, 0)$ is the Hamiltonian vector field generated by the Poisson bracket of the system for the real-valued function $-A$. Controllability of this system and the possibility of stabilizing unstable perturbations of the Föppl equilibrium are some of the problems being currently considered.

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References


