Keywords: control of flexion torsion loaded bodies, robustness, Infinite dimensional systems, eigenfrequencies invariance

Abstract

We present here in the framework of resonant behaviors an eigenfrequency invariance properties for loaded flexural and torsional deformable bodies. We found that under a preload condition each eigenfrequency is invariant versus the tipload $m_N$. For the mode 7 and beyond the preload conditions are less than $m_N = 1$, and the seven first modes (form 0 to 6) can be easily modelled by polynomial functions of $m_N$ in the neighbourhood of useful $m_N$. To further improve this result, the role of passive system parameters has been studied. We will show that compliant behaviors at mechanical joints create a new order on the system which leads to a large invariance property in the $m_N$ space. Compliant mechanical joint diminishes drastically the preload condition, as shows a lower threshold value for $m_N$ and a lower invariance mode number (now typically 3) moreover the invariance is more precise. These properties open the way to new finite dimension controllers.

1 Introduction

The research of higher performances in speed and precision for industrial robotised mechanical systems has been leading to lightweight mechanical structures [1, 3]. Such structures cannot be maintained as rigid bodies and deformations take place which considerably modify their behavior. As the system becomes now infinite dimensional, the control to be applied for stabilizing its trajectory becomes itself infinite dimensional, which fundamentally changes its nature. First sensing the modes and computing the corresponding control rapidly reaches the limits of existing technology, and second actuators themselves cannot deliver the required power in the high frequency range corresponding to these modes. Several approaches of control exist to manage and control infinite dimensional structures [4], but they are fragilized by the notion of trajectory [5], or by a non appropriate diskretized state space approach [6]. It is very important to research a way to approach differently the problem. In a first step, it is interesting to identify system properties allowing to determine in which cases, if any, the infinite dimensional deformable system reduces to a finite dimensional one controlled by a finite dimensional controller. Here specializing to the most penalizing case of initially rigid one link system, it is shown that past a fixed order (typically 7) the modes are totally invariant as a function of the applied tip load, and for lower order ($< 7$), are exhibiting a smooth behavior easily modelled by simple rational function of the load. Preliminary analysis of flexion deformation resonance modes of one link system was showing interesting invariance properties of resonance frequencies versus applied tip load, allowing to construct finite dimensional controllers with arbitrary prescribed preciseness. Moreover to further improve this result, the role of passive system parameters has been studied. It is shown that addition of compliance effect at link origin changes the structure of equations as a consequence of modification of power distribution between resonants modes. In this enlarged system, non compliant mode structure appears to be singular. The most significant results of this compliant approach are that there exits a threshold compliance value above which all resonance frequencies are invariant versus the applied tip load. This new invariance has not the same behavior as in the non compliant case, as shows a lower threshold value.

2 Preliminary results: the only flexion approach

Let $\mathcal{L}$ be the Lagrangian of motion of linearized deformation equations of a one degree of freedom robotic arm [7, 10],

$$\mathcal{L} = \int_0^T \int_0^L \left\{ \left( \frac{1}{2} \chi (x \dot{\theta} + \ddot{u})^2 - \left( \frac{1}{2} \alpha u^2 \right) \right) \right\} \, dx \, dt \quad (1)$$

where $u \equiv u(x,t)$ is the first order flexible deformation function in meter, $\theta$ the arm angular position in radian, and $\ddot{u}$ or $\dot{\theta}$ and $u''$ are respectively the first time derivative of $u$ or $\theta$ and the second space derivative of $u$, $L$ is the length of
flexible arm, and \( \alpha \) and \( \chi \) are appropriate mechanical constants [11, 12]. The study of resonant equilibrium state, i.e. \( \theta = \text{cste} \) with arbitrary initial conditions on \( u(x, t) \), leads to the following solution:

\[
u(x, t) = v(x) \cos(\omega t)
\]

(2)

with \( v(x) = a \cosh(Kx) + b \sinh(Kx) + c \cos(Kx) + d \sin(Kx) \) and \( (a, b, c, d) \in \mathbb{R}^4 \). Adding the associated boundary conditions of the problem, i.e. \( u(0, t) = u'(0, t) = u''(L, t) = 0 \) and \( u'''(L, t) = M \ddot{u}(L, t) \), for all \( t \in [0, T] \), where \( M \) is a punctual mass at \( x = L \), the following equation of degeneration associated to system eigenmodes is found:

\[
K^2(1 + \cos(KL) \cosh(KL)) = -\frac{M \omega^2}{\alpha} (\cos(KL) \sinh(KL) - \sin(KL) \cosh(KL))
\]

(3)

with \( K = (\chi \omega^2/\alpha)^{1/4} \). It can be reduced to the following normalized form,

\[
(1 + \cos(\omega_N) \cos(\omega_N)) = -m_N \omega_N \cos(\omega_N) \sinh(\omega_N) - \sin(\omega(\omega_N) \cosh(\omega_N))
\]

(4)

with \( \omega_N = KL, m_N = M/\chi L \alpha \), and \( \chi L \) the weight of flexible arm. The small deformation assumption used to derive (3), i.e. that the curvilinear abscissa along deformed arm \( s \) is equivalent to \( s \) [13], the abcissa along rigid arm, leads to the following condition,

\[
\frac{\omega^2}{\omega_f^2} \leq 1/3
\]

(5)

with \( \omega_f^2 = 1/m_N L^4 \chi, \) and \( \omega_f = \hat{\theta} \). Results of the computation are however shown over the complete interval to point out the regularity of the solution. Consider the behavior of eigenfrequencies of \( u \) vs load \( m_N \) at flexible arm tip. Let \( k \) be the mode index, it can easily be shown that asymptotically,

\[
\lim_{m_N \to +\infty} \omega_N(m_N, 0) = 0
\]

\[
\lim_{m_N \to +\infty} \omega_N(m_N, k) = \pi/4 + k \pi
\]

(6)

see figure 1. The convergence rate of mode \( k, k \neq 0 \), grows with \( k \), and refering to figure 1 for \( m_N \geq 2 \), \( \omega_N(m_N, k) = \pi/4 + k \pi \). This pseudo invariance property opens the way to finite dimensional intelligent controller because under any preload constraint \( m_N \geq 2 \), the infinite subset of eigenfrequencies and eigenvectors \( k > 2 \) can be predefined numerically independently of subsequent motion.

3 Preliminary results: the only flexion approach with compliance

With same Lagrangian of motion as in (1), the same resonant equilibrium state leads to same solution \( u(x, t) = v(x) \cos(\omega t) \) with \( v(x) = a \cosh(Kx) + b \sinh(Kx) + c \cos(Kx) + d \sin(Kx) \), but with new associated boundary conditions,

\[
\begin{align*}
u(0, t) &= 0, \quad \lambda_0 L u''(0, t) + (1 - \lambda_0) u'(0, t) = 0 \\
\lambda_L L u'''(L, t) + (1 - \lambda_L) u''(L, t) &= 0
\end{align*}
\]

(7)

\[
u'''(L, t) = M \ddot{u}(L, t)
\]

for all \( t \in [0, T] \). \( \lambda_L \) and \( \lambda_0 \) are the compliance constants respectively at \( x = L \) and \( x = 0 \). \( \lambda_0 \) and \( \lambda_L \in [0, 1] \).

For \( \lambda_0 = 0 \) we have the classical rigid boundary condition \( u'(0, t) = 0 \) \( \forall t \) and for \( \lambda_L = 1 \) we have the free link condition \( u''(0, t) = 0 \) \( \forall t \). The same conclusion holds for \( \lambda_L \).

The following equation of degeneration associated to system eigenmodes is now found:

\[
(1 - \lambda_L)(1 - \lambda_0)K^3(1 + \cos(KL) \cosh(KL)) + (1 - \lambda_L) \lambda_0 L K^2 (\sin(KL) \cosh(KL) - \cos(KL) \sinh(KL)) =
\]

\[
-\frac{M \omega^2}{\alpha} \begin{cases} &[(1 - \lambda_L)(1 - \lambda_0) \cos(KL) \sinh(KL) - \sin(KL) \cosh(KL) + \sin(KL) \cosh(KL) + \cos(KL) \sinh(KL) + \lambda_L(1 - \lambda_0) \sin(KL) \cosh(KL)] 
\end{cases}
\]

(8)

with \( K = (\chi \omega^2/\alpha)^{1/4} \). It reduces to the following normalized form,

\[
(1 - \lambda_L)(1 - \lambda_0)(1 + \cos(\omega_N) \cos(\omega_N)) + (1 - \lambda_L) \lambda_0 \omega_N(\sin(\omega_N) \cosh(\omega_N) - \cos(\omega_N) \sinh(\omega_N)) =
\]
Figure 2: mode $k = 0$

Figure 3: We present here for the mode $k = 0$, figure 2 and for the mode $k = 1$, figure 3, the curves $\omega_{Nk}(m_N, \lambda)$, for $(m_N, \lambda) \in [0, 10] \times [0, 1]$. We can see easily a large domain of invariance which is called the compliant behavior.

Figure 4: mode $k = 4$

Figure 5: We present here for the mode $k = 4$, figure 4 and for the mode $k = 5$, figure 5, the curves $\omega_{Nk}(m_N, \lambda)$, for $(m_N, \lambda) \in [0, 10] \times [0, 1]$. We can see easily a large domain of invariance which is called the compliant behavior.

Figure 6: The first 6 eigenmodes $\omega_{Nk}, k \in [0, 5]$ from down to up, versus the normalized load $m_N$ at $\lambda=0$, i.e. in the non-compliant behavior.

Figure 7: The first 6 eigenmodes $\omega_{Nk}, k \in [0, 5]$ from down to up, versus the normalized load $m_N$ at $\lambda=1$, i.e. in the compliant behavior, we can easily see a very soon and very precise invariance effect, even for the mode $k = 0$. 
\[-m_N\omega_N \{ (1 - \lambda_L)(1 - \lambda_0)(\cos(\omega_N) \sinh(\omega_N) - \\
\sin(\omega_N) \cosh(\omega_N) \\
+ \omega_N(1 - \lambda_L) \sinh(\omega_N) \cos(\omega_N) + \sin(\omega_N) \cosh(\omega_N)) \\
+ \omega_N (2(1 - \lambda_L) - \lambda_L(1 - \lambda_0)) \sin(\omega_N) \sinh(\omega_N) \}\] (9)

with \(\omega_N = KL, m_N = M/(\chi L),\) and \(\chi L\) the weight of flexible arm. For \(\lambda_0 = \lambda_L = \lambda,\) figures 2 to 5 show \(\omega_N(m_N, k)\) versus \((m_N \times \lambda)\in[0, 10] \times [0, 1].\) Apart small areas of transition these surfaces are very flat, meaning that there are typically two main behaviors; the non compliant behaviour close to \(\lambda = 0,\) and the compliant behaviour for \(\lambda\) close to 1. The frontier between these two behaviours becomes very small and tends to \(\lambda = 0\) as \(k\) increases. In the compliant behavior

\[\omega_N(m_N, k) = 3\pi/4 + k\tau\text{ for } m_N \geq 1/2 \] (10)

see figures 2 to 5. The convergence rate versus \(m_N\) is increased, and the reduction of preload constraint to \(m_N \geq 0.5,\) shows an increasing more efficient and more precise invariance effect in compliant case, as shown by comparing figure 6 to figure 7. We can also noticed that the compliant behavior can be reach for \(\lambda_0 = 0.3,\) i.e. for realistic technological devices.

4 Preliminary results: The only flexion approach in dissipative and non compliant case

Let \(\mathcal{L}\) be the Lagrangian of motion of the linearized equation of deformation for the system defined Figure 1 in the dissipative case.

\[\mathcal{L} = \int_0^T \int_0^L \left( \frac{1}{2} \chi(x\dot{u} + \ddot{u})^2 + \left( \frac{1}{2} D(t) - \frac{1}{2} \alpha \nu^2 \right) dx \right) dt\]

two cases of dissipation are considered here, first the viscous case then \(D(t) = \beta \int_0^t \ddot{u}^2 d\zeta,\) secondly the Voigt dissipation case then \(D(t) = \beta \int_0^t \ddot{u}^2 d\zeta.\) This leads to the two normalized differential equations

\[\frac{\partial^2 u}{\partial \tau^2} + \beta_N \frac{\partial u}{\partial \tau} + \frac{\partial^4 u}{\partial x^4} = -x \frac{\partial^2 \theta}{\partial \tau^2}\]

\[\frac{\partial^2 u}{\partial \tau^2} + \beta_N \frac{\partial u}{\partial \tau} + \frac{\partial^4 u}{\partial x^4} = -x \frac{\partial^2 \theta}{\partial \tau^2}\]

involved respectively in the viscous and Voigt dissipation cases, with \(\tau = t\omega, \omega = \sqrt{\frac{1}{\alpha}}, \beta_N = \sqrt{\frac{\alpha}{\beta}}\) and \(\omega_N = \omega/\omega_1.\) The study of the resonant equilibrium state leads to the following solution, \(u(x, t) = v(x) \cos(\omega_N \tau) + w(x) \sin(\omega_N \tau)\) with \(v(x) = a \cos(Kx) + b \sin(Kx) + c \cos(Kx) + d \sin(Kx),\) and \(w(x) = \bar{a} \cos(Kx) + \bar{b} \sin(Kx) + \bar{c} \cos(Kx) + \bar{d} \sin(Kx).\) Adding to the boundary conditions of the problem, i.e. \(v(0) = v'(0) = v''(L) = 0\) and \(w''(L) = -(M/\alpha) \dot{\omega}(L) = 0\) and \(w'(0) = w''(0) = w''(L) = -\alpha \omega^2 \omega'(L),\) we found the following equations of degeneration associated to the eigenmodes of the system.

\[(A^3 - 3AB^2)(4 + 2\cos((A + B)L)\cosh((A - B)L) + 2\cos((A - B)L)\cosh((A + B)L) - (3A^2B - B^3)\]

\[(2\sin((A + B)L)\sinh((A - B)L) + 2\sin((A - B)L)\sinh((A + B)L) - 2\sin((A + B)L) \cosh((A - B)L) + 2\cos((A + B)L) \cosh((A + B)L) + 2\sin((A - B)L) \sinh((A - B)L) + 2\cos((A + B)L) \sinh((A + B)L)\]

with \((A + iB)^4 = \omega_N^2 + i\beta_N \omega_N\) for the first type of dissipation (viscous case) and with \((A + iB)^4 = \omega_N^2(1 + \beta_N^2)/(1 + \beta_N^2)\) for the Voigt dissipation case. For the generally used materials, \(\beta_N\) is very small, approximatively \(10^{-2}.\) We have shown that for this kind of value the dissipation do not modify the previously shown behaviours. Moreover we have also shown that in the case of dissipative and compliant motion, the associated complex equation of degeneration which is left here for reason of conicseness, do not modify the previously shown behaviors.

5 The flexion-torsion equation of degeneration for loaded bodies

Applying Lagrangian Formalism the complete equations of one link deformable mechanical system, is in the following form.

\[\rho A(x \frac{d^2 \theta}{dt^2} + \frac{\partial^2 u(t, x)}{\partial t^2}) = \frac{\partial^2}{\partial x^2} \frac{\partial^2 u(t, x)}{\partial x^2}\]

\[\rho K \frac{\partial^2 \gamma(t, x)}{\partial t^2} = \frac{\partial}{\partial x} \frac{GJ \gamma(t, x)}{\partial x}\]

\[m \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial x} \frac{EI \partial^2 u(t, x)}{\partial x^2}\]

\[m \frac{\partial^2 X}{\partial t^2} + J_f(x \frac{d^2 \theta}{dt^2} + \frac{\partial^3 u(t, x)}{\partial x^3}) \bigg|_{x = L} = -EI \frac{\partial^2 u(t, x)}{\partial x^2} \bigg|_{x = L}\]
Figure 8: mode $k = 0$, $\omega_{N0} = f(m_N)$, $m_N \in [0, 1000]$

Figure 9: mode $k = 1$, $\omega_{N1} = f(m_N)$, $m_N \in [0, 1000]$

Figure 10: mode $k = 3$, $\omega_{N3} = f(m_N)$, $m_N \in [0, 100]$

Figure 11: mode $k = 4$, $\omega_{N4} = f(m_N)$, $m_N \in [0, 10]$

Figure 12: mode $k = 6$, $\omega_{N6} = f(m_N)$, $m_N \in [0, 10]$

Figure 13: mode $k = 7$, $\omega_{N7} = f(m_N)$, $m_N \in [0, 10]$
Figure 14: mode $k = 0$, $\omega_{N0} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 1000] \times [0, 1]$

Figure 15: mode $k = 1$, $\omega_{N1} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 400] \times [0, 1]$

Figure 16: mode $k = 2$, $\omega_{N2} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 80] \times [0, 1]$

Figure 17: mode $k = 3$, $\omega_{N3} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 40] \times [0, 1]$

Figure 18: mode $k = 5$, $\omega_{N5} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 10] \times [0, 1]$

Figure 19: mode $k = 6$, $\omega_{N6} = f(m_N, \lambda_0)$, $(m_N \times \lambda_0) \in [0, 5] \times [0, 1]$
\[
ml \frac{\partial^2 X}{\partial t^2} + J_l \frac{\partial^2 \gamma(x,t)}{\partial t^2} \bigg|_{x=L} = -GJ \frac{\partial \gamma(x,t)}{\partial x} \bigg|_{x=L}
\]

(15)

with \( \theta, u(x,t), \gamma(x,t) \) respectively the articular variable, and the deformation, flexion and torsion, variables, \((J_l, L)\) the coordinates of the tip mass \( m \) with respect to the end of the link, \( L \) is the length of the body, and the various other coefficients characterizing the beam are as usual within Euler-Bernouilli approximation [2]. Boundary Conditions are given by eqns(13,14,15), and,

\[
X = (L + l_f) \theta + l_f \frac{\partial u(x,t)}{\partial x} \bigg|_{x=L} + l_f \gamma(x,t) \bigg|_{x=L} + u(x,t) \bigg|_{x=L}
\]

(16)

We want to study the resonant behaviour of this non dissipative system, and we pose for that, \( \theta = 0 \), we suppose also for reasons of simplicity that \( EI \) and \( GJ \) do not depend on \( x \). By assuming that, \( \zeta = x/L, \lambda_1 = l_f/L, \lambda_2 = l_f/L, \hat{u} = u/L, \omega_1^2 = EI/\rho A L^3, \omega_2^2 = GJ/\rho k^2 L^4, \omega_2 = \omega_1/\lambda_1, \lambda_m = K^2/2L^4, m_N = m/\rho A L^3, J_N = J_f/\rho A L^3, J_N = J_f/\rho k^2 L^4, \tau \), we found the following normalized equations of motion,

\[
\frac{\partial^2 \hat{u}}{\partial \zeta^2} + \frac{\partial^4 \hat{u}}{\partial \zeta^4} = 0
\]

(17)

\[
\frac{\partial^2 \gamma}{\partial \zeta^2} - \frac{1}{\lambda_2^2} \frac{\partial^2 \gamma}{\partial \zeta^2} = 0
\]

(18)

with, for all \( \tau \), the following normalized boundary conditions,

\[
\hat{u}(0,\tau) = 0 \quad (19), \quad \hat{u}'(0,\tau) \equiv \frac{\partial \hat{u}}{\partial \zeta} \bigg|_{\zeta=0} = 0 \quad (20), \quad \gamma(0,\tau) = 0
\]

(21)

\[
\hat{u}''(1,\tau) = \left. \frac{\partial^2 \hat{u}}{\partial \zeta^2} \right|_{\zeta=1} = -\lambda_f \hat{u}''(1,\tau) - J_{IN} \frac{\partial^3 \hat{u}}{\partial \tau^3} \bigg|_{\zeta=1}
\]

(22)

\[
\hat{u}'''(1,\tau) = \left. \frac{\partial^3 \hat{u}}{\partial \zeta^3} \right|_{\zeta=1} = m_N \left( \frac{\partial^2 \hat{u}}{\partial \tau^2} + \lambda_1 \frac{\partial^3 \hat{u}}{\partial \zeta^3} + \lambda_2 \frac{\partial^2 \gamma}{\partial \zeta^2} \right) \bigg|_{\zeta=1}
\]

(23)

\[
(\frac{1}{\lambda_2})^2 \hat{\gamma}''(1,\tau) = \left( \frac{1}{\lambda_2} \right)^2 \hat{\gamma}''(1,\tau) - \left( \frac{1}{\lambda_2^2} \right) \frac{\partial \hat{\gamma}''(1,\tau)}{\partial \zeta} - J_N \frac{\partial^2 \gamma}{\partial \tau^2} \bigg|_{\zeta=1}
\]

(24)

In the case of stationary resonant behaviours \( \hat{u}(\zeta, \tau) \), solution of (17), can be written as,

\[
\hat{u}(\zeta, \tau) = \hat{u}_1(\zeta) \cos(\omega_N \tau)
\]

(25)

with \( \omega_N = \omega_{1/2} \), where \( \omega \) is the frequency variable, and,

\[
\hat{u}_1(\zeta) = a \cosh(\omega_{1/2} \zeta) + b \sinh(\omega_{1/2} \zeta)
\]

(26)

\[
\gamma(\zeta, \tau), \text{solution of } (8), \text{can be written as,}
\]

\[
\gamma(\zeta, \tau) = \gamma_1(\zeta) \cos(\omega_N \tau)
\]

(27)

with,

\[
\gamma_1(\zeta) = \hat{a} \cos(\omega_N \lambda_2 \zeta) + \hat{b} \sin(\omega_N \lambda_2 \zeta)
\]

(28)

Introducing (25) and (27) into the six boundary conditions of the problem, equations (19) to (24), leads by the elimination of \( \cos(\omega_N \tau) \) to a set of 6 linear equations of the six initial values \((a, b, c, d, \hat{a}, \hat{b})\). While this linear system is degenerated the only condition of existence of resonant mode is that the determinant of this linear system equals zero. This leads to the following equation of degeneration,

\[
\cos(\omega_N \lambda_2) \left( -\frac{2\omega_N^4}{\lambda_2} - \frac{2J_{IN} m_N \omega_N^6}{\lambda_2} \right) + \sin(\omega_N \lambda_2) \left( 2J_{IN} \omega_N^5 + \frac{2\lambda_2^2 \omega_N^5}{\lambda_m} + 2J_{IN} J_{IN} m_N \omega_N^7 \right) + \cos(\omega_N^{1/2} - \lambda_2 \omega_N) \cosh(\omega_N^{1/2})
\]

\[
\left( -\frac{\omega_N^2}{\lambda_2} - J_{IN} m_N \omega_N^{11/2} + \frac{J_{IN} m_N \omega_N^6}{\lambda_2} - J_{IN} J_{IN} \omega_N^{13/2} \right) + \sin(\omega_N^{1/2} - \lambda_2 \omega_N) \sinh(\omega_N^{1/2}) \left( \frac{\lambda_1 \omega_N^5}{\lambda_2} + \frac{\lambda_1 m_N \omega_N^5}{\lambda_2} - J_{IN} J_{IN} \omega_N^{13/2} + \frac{J_{IN} \lambda_2^2 \omega_N^{13/2}}{\lambda_m} \right)
\]

\[
+ \cos(\omega_N^{1/2} + \lambda_2 \omega_N) \cosh(\omega_N^{1/2}) \left( -\frac{\omega_N^4}{\lambda_2} + \frac{J_{IN} m_N \omega_N^{11/2}}{\lambda_2} + \frac{J_{IN} \lambda_1^2 \omega_N^{13/2}}{\lambda_m} \right) + \sin(\omega_N^{1/2} + \lambda_2 \omega_N) \sinh(\omega_N^{1/2}) \left( \frac{\lambda_1 \omega_N^5}{\lambda_2} + \frac{\lambda_1 m_N \omega_N^5}{\lambda_2} - J_{IN} J_{IN} \omega_N^{13/2} + \frac{J_{IN} \lambda_2^2 \omega_N^{13/2}}{\lambda_m} \right)
\]

\[
+ \cos(\omega_N^{1/2} - \lambda_2 \omega_N) \cosh(\omega_N^{1/2}) \left( -\frac{\omega_N^4}{\lambda_2} - \frac{J_{IN} m_N \omega_N^{11/2}}{\lambda_2} - \frac{J_{IN} \lambda_1^2 \omega_N^{13/2}}{\lambda_m} \right) + \sin(\omega_N^{1/2} - \lambda_2 \omega_N) \sinh(\omega_N^{1/2}) \left( \frac{\lambda_1 \omega_N^5}{\lambda_2} - \frac{J_{IN} \lambda_2^2 \omega_N^{13/2}}{\lambda_m} \right)
\]
of the eigenfrequencies, their position in the spectrum of the useful mode is separated from his neighbour by the preload condition. We see that as predicted theoretically, every mode after preload condition converges to a constant value (in variance). For the second case we have, $\omega_{Nk}^2 = \frac{\omega_N^2}{\lambda_\omega} \left( \frac{2 \omega_N^2}{\lambda_m} + \frac{2 J_F \omega_N^2}{\lambda_\omega} \lambda_N \omega_N^0 \right) + \frac{k \pi}{\lambda_\omega}$, where $k, k \in \mathbb{N}$, is the mode index, for large values of $\omega_N$ we have,

$\omega_{Nk} = \frac{k \pi}{\lambda_\omega}$

Secondly the case $m_N = 0$ leads for large $\omega_N$ to the equation,

$\omega_{Nk} = \frac{\omega_N}{\lambda_\omega} \left( \frac{2 \omega_N^2}{\lambda_m} + \frac{2 J_F \omega_N^2}{\lambda_\omega} \lambda_N \omega_N^0 \right) + \frac{k \pi}{\lambda_\omega}$

i.e. for large values of $\omega_N$,

$\omega_N = \frac{k \pi}{\lambda_\omega}$

We found then that $\omega_N = f(m_N)$ has a constant behaviour (invariance property) for large values of $m_N$, and that each mode is separated from his neighbour by $\Delta \omega_N = \pi / \lambda_\omega$. We see also that for large values of $\omega_N$ this invariance property do not depends on the geometry of the system but only on $\lambda_\omega$. For a more precise approach of these phenomena we have computed using (29) the curves $\omega_N = f(m_N)$ for the several one of the eight first modes see figure 8 to 13. We see that as predicted theoretically every mode after preload condition converges to a constant value (invariance). For the mode $k = 0$ this mode converges to 0 for only $m_N > 10000$ not presented in our figure. We see also that for $k \geq 7$ the preload condition is less than $m_N = 1$, so the infinite subset of the eigenfrequencies $k \in [7, +\infty]$ can be represented by precalculated constants opening the way to new finite dimension controllers. The seven first mode $k \in [0, 6]$ can be easily modelled by polynomial functions of $m_N$ in the neighbourhood of the usefulness $m_N$.

7 The flexion-torsion equation of degeneration in the compliant case

If we consider as in the preceeding sections the dynamical equation of a one link deformable mechanical system, with the same notation and with the same normalized variables, we found the following normalized equations of motion,

$\frac{\partial^2 \bar{u}}{\partial \tau^2} + \frac{\partial^4 \bar{u}}{\partial \zeta^4} = 0$ (30)

$\frac{\partial^2 \gamma}{\partial \tau^2} - \left( \frac{1}{\lambda_\omega} \right)^2 \frac{\partial^2 \gamma}{\partial \zeta^2} = 0$ (31)

with, for all $\tau$, the following normalized boundary conditions,

$\bar{u}(0, \tau) = 0$ (32), $\gamma(0, \tau) = 0$ (33)

$\lambda_0 \bar{u}''(0, \tau) + (1 - \lambda_0) \bar{u}'(0, \tau) \equiv \lambda_0 \frac{\partial^2 \bar{u}}{\partial \zeta^2} \bigg|_{\zeta = 0} + (1 - \lambda_0) \frac{\partial \bar{u}}{\partial \zeta} \bigg|_{\zeta = 0} = 0$ (34)

$\bar{u}''(1, \tau) \equiv \frac{\partial^2 \bar{u}}{\partial \zeta^2} \bigg|_{\zeta = 1} = -\lambda_\tau \bar{u}''(1, \tau) - J_{\tau N} \frac{\partial^3 \bar{u}}{\partial \tau^2 \partial \zeta} \bigg|_{\zeta = 1}$ (35)

$u'''(1, \tau) \equiv \frac{\partial^3 \bar{u}}{\partial \zeta^3} \bigg|_{\zeta = 1} \equiv m_N \left( \frac{\partial^2 \bar{u}}{\partial \tau^2} + \lambda_\tau \frac{\partial^3 \bar{u}}{\partial \tau^2 \partial \zeta} + \lambda_l \frac{\partial^2 \gamma}{\partial \tau^2} \right) \bigg|_{\zeta = 1}$ (36)

$\left( \frac{1}{\lambda_\omega} \right)^2 \frac{\partial^2 \gamma}{\partial \zeta^2} \bigg|_{\zeta = 1} = \left( \frac{1}{\lambda_\omega} \right)^2 \frac{\partial \gamma}{\partial \zeta} \bigg|_{\zeta = 1} - \lambda_\lambda u'''(1, \tau) - J_{\tau N} \frac{\partial^2 \gamma}{\partial \tau^2} \bigg|_{\zeta = 1}$ (37)

Equation (34) is the compliant boundary condition that we supposed to be applied only on $\zeta = 0$, $\lambda_0$ is the compliant parameter, $\lambda \in [0, 1]$, $\lambda = 0$ no compliance, $\lambda = 1$ full compliance. In the case of stationary resonant behaviors $\bar{u}(\zeta, \tau)$, solution of (30), can be written as,

$\bar{u}(\zeta, \tau) = \bar{u}_1(\zeta) \cos(\omega_N \tau)$ (38)

with $\omega_N = \omega / \omega_j$, where $\omega$ is the frequency variable, and,

$\bar{u}_1(\zeta) = a \cosh(\omega_N^2 \zeta) + b \sinh(\omega_N^2 \zeta)$

$+ c \sin(\omega_N^2 \zeta) + d \cos(\omega_N^2 \zeta)$ (39)

$\gamma(\zeta, \tau)$, solution of (2), can be written as,

$\gamma(\zeta, \tau) = \gamma_1(\zeta) \cos(\omega_N \tau)$ (40)

with,

$\gamma_1(\zeta) = \bar{a} \cos(\omega_N \lambda_\omega \zeta) + \bar{b} \sin(\omega_N \lambda_\omega \zeta)$ (41)
Introducing (38) and (40) into the six boundary conditions of the problem, equations (32) to (37), leads by the elimination of \( \cos(\omega_N N) \) to a set of 6 linear equations of the six initial values \((a, b, c, d, \tilde{a}, \tilde{b})\). While this linear system is degenerated, the only condition of existence of the resonant mode is determinant of this linear system equals zero. This leads to the following equation of degeneration,

\[
\frac{F_1(\omega N)}{\lambda_w} (1-\lambda_0) \omega_N^4 (-1 + \lambda_f \omega_N (1 - m_N) - J_{I N M N} \omega_N^2) + F_2(\omega N) \omega_N^3 (2(1 - \lambda_0) (1 - m_N) + 4\lambda_0 J_{I N M N}) \]

\[
+ \frac{F_3(\omega N)}{\lambda_w} (1-\lambda_0) \omega_N^2 (1 - 1 - \lambda_f \omega_N (1 - m_N)) + J_{I N M N} \omega_N^2)
+ F_4(\omega N) \omega_N (2(1 - \lambda_0) (1 - \lambda_f \omega_N (1 - m_N) + J_{I N M N}) \omega_N^2)
+ F_5(\omega N) \omega_N (2(1 - \lambda_0) (1 - \lambda_f \omega_N (1 - m_N) + J_{I N M N}) \omega_N^2)
+ F_6(\omega N) \omega_N (2(1 - \lambda_0) (1 - \lambda_f \omega_N (1 - m_N) + J_{I N M N}) \omega_N^2)
+ F_7(\omega N) \omega_N (2(1 - \lambda_0) (1 - \lambda_f \omega_N (1 - m_N) + J_{I N M N}) \omega_N^2)
\]

with

\[
F_1(\omega N) = \cos^2(\omega_N^2) \cos(\lambda_w \omega_N)
\]

\[
F_2(\omega N) = \cos(\omega_N^2) \cos(\lambda_w \omega_N) \cos h(\omega_N^2)
\]

\[
F_3(\omega N) = \cos^2(\omega_N^2) \cos(\lambda_w \omega_N)
\]

\[
F_4(\omega N) = \cos(\omega_N^2) \cos(\lambda_w \omega_N) \sin(\omega_N^2)
\]

\[
F_5(\omega N) = \sin^2(\omega_N^2) \cos(\lambda_w \omega_N)
\]

\[
F_6(\omega N) = \cos^2(\omega_N^2) \sin(\lambda_w \omega_N)
\]

\[
F_7(\omega N) = \cos(\omega_N^2) \sin(\lambda_w \omega_N) \cos h(\omega_N^2)
\]

\[
F_8(\omega N) = \sin^2(\omega_N^2) \sin(\lambda_w \omega_N)
\]

\[
F_9(\omega N) = \cos(\omega_N^2) \sin h(\omega_N^2)
\]

\[
F_{10}(\omega N) = \sin^2(\omega_N^2) \sin h(\omega_N^2)
\]

\[
F_{11}(\omega N) = \cos(\omega_N^2) \cos(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

\[
F_{12}(\omega N) = \sin(\omega_N^2) \cos(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

\[
F_{13}(\omega N) = \cos(\omega_N^2) \sin(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

\[
F_{14}(\omega N) = \sin(\omega_N^2) \sin(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

\[
F_{15}(\omega N) = \cos(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

\[
F_{16}(\omega N) = \sin(\lambda_w \omega_N) \sin h(\omega_N^2)
\]

Equation (42) gives implicitly, for a given geometry, the eigenfrequencies \( \omega_N \) versus the load \( m_N \), thus permitting the study of the resonant mode versus the load.

8 The flesion-torsion invariance properties in the compliant case

The study of (42) in the compliant case \( \lambda_0 = 1 \) and in the limit of large \( \omega_N \) gives two interesting results. First the case of large \( \omega_N \) leads to the equation,

\[
\omega_{Nk}^{1/2} = Artg \left( \frac{1}{m_N} \right) + k\pi
\]

where \( k \) is the mode index, when \( m_N \to 0 \),

\[
\omega_{Nk}^{1/2} = \frac{\pi}{2} + k\pi
\]

when \( m_N \to +\infty \).

\[
\omega_{Nk}^{1/2} = k\pi
\]

Secondly we see that these asymptotic behaviors are different as the non compliant case. Here \( \Delta \omega_N = \pi^2 \) in place
of $\pi/\lambda_0$ and the analytical approach of (29) show that the equation of compliance are completely different in its structure as the non compliant case. For a more precise approach of these phenomena we have computed using (42) the curves $\omega_{mN} = f(m_N, \lambda_0)$, where $\lambda_0$ is the compliant variable, for several one of the eight first modes see figure 14 to 19. We see as predicted theoretically that in the $(\omega_{mN}, m_N, \lambda_0)$ space there are mainly two behaviors, the non compliant one which is singular and defined for $\lambda_0 < 0.3$ and the compliant one which stays in the rest of the space and leads to a large invariant behaviour. In the framework of the compliant behavior we see easily even for the mode $k = 0$ that the threshold value of invariance is drastically reduced and that the invariance is more precise and stable than without compliance. For example the threshold condition for the mode $k = 6$ is with compliance $m_N = 0.5$ and $m_N = 2$ without compliance. We can also noticed that for reason of complexity of the problem, in contrary of the only flexion approach where we have introduce compliance at the two limits of the arm, i.e. on $x = 0$ and on $x = L$, in flexion-torsion approach we have only introduce compliance at $x = 0$, this explain at least in part, the less efficient effect of the compliance on the flexion-torsion approach compared to the only flexion approach.

9 Conclusion

Analysis of resonant solutions of one link deformable system shows that, as a function of the applied tip load, the resonant frequencies are distributed in equally spaced layers splitted by a distance $\pi \omega_1/\omega_f$, where $\omega_1$ and $\omega_f$ are the characteristic torsion and flexion frequencies of the system. Furthermore, the modes of order $k > 7$ are typically converging to the layer boundary $k \pi \omega_1/\omega_f$ for normalized tip load equal to 1, and have for $k < 7$ a very smooth behavior representable by simple rational fraction of tip load, allowing to easily predict the location of resonances for well defined system parameters, and to determine the domains in which they are invariant, a property which will be used for the design of finite dimension control. Moreover analysis of compliance effect on deformable system shows high sensitivity of the system to this parameter because of modification of power distribution over resonant modes. It is shown that there exists a threshold compliance value ($\lambda_0 = 0.3$) above which resonance frequencies are invariant versus tip load and are independent of mode number. This threshold value is easily technically realizable as it corresponds to a current value of axial spring stiffness representing the compliance. A further improvement has been observed when adding compliance at link tip. This observation would seem to oppose to common idea that higher stiffness is required when the system becomes more deformable. In fact existence of compliance power sink allows better internal regulation of power distribution over the resonant modes, which will not have to be taken care off by an outer controller.

References