Analysis of a Nonlinear Dynamical Model of an Axial Dispersion Nonisothermal Reactor

M. E. Achhab†, M. Laabissi†, J. J. Winkin△∗, D. Dochain‡

†Université Chouaib Doukkali, Faculté des Sciences - BP 20, El Jadida, Morocco
Fax: +212-3-342187
achhab@ucd.ac.ma

△University of Namur (FUNPD), Department of Mathematics,
8 Rempart de la Vierge, B-5000 Namur, Belgium
Fax: +32-81-724914, Tel: +32-81-724910
Joseph.Winkin@fundp.ac.be

‡CESAME, UCL, 4-6 avenue G. Lemaître, B-1348 Louvain-la-Neuve, Belgium
Fax: +32-10-472180
dochain@csam.ucl.ac.be

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1 Introduction

The dynamics of tubular reactors are typically described by nonlinear partial differential equations (PDE), derived from mass and energy balance laws, see e.g. [4]. In the case of isothermal reactions, the dynamical analysis of the linearized tangent model of such systems has been carried out in [9], by using a $C_0$-semigroup Hilbert state-space formulation. Axial dispersion reactors and plug flow reactors have been studied there. However if the objective is e.g. to control the process, depending on the type of reactions, the nonlinearities may be such that they can not be neglected without a serious deterioration of the desired behavior of the system. It is then important to account for such nonlinearities as much as possible, especially in the case of nonisothermal reactors, where in addition the PDE’s may be highly coupled. This paper is dedicated to the dynamical analysis of a nonisothermal axial dispersion reactor using a nonlinear model. The state trajectories of the latter are studied by taking the nonlinear terms explicitly into account. The existence and the uniqueness of the temperature and reactant concentration profile trajectories are proved on the whole time axis. Our approach is based on nonlinear functional analysis tools, as developed e.g in [5]. It is comparable to the one carried out in [1] for plug flow reactors. In particular, in the main result reported here, Lipschitz and dissipativity properties of the nonlinear operator involved in the dynamics generator are considered. This result is also based on the invariance of the domain of the nonlinear operator with respect to the $C_0$-semigroup generated by the linear part of the system. Note that a related nonisothermal reactor model has been considered in [3]. This paper is organized as follows. In Section 2 we introduce the nonlinear dynamical model of the axial dispersion nonisothermal reactor and give a precise statement of the problem. Then in Section 3 a useful result from nonlinear functional analysis is recalled. Section 4 gives some technical lemmas and the main result is reported in Section 5. Some technical calculations are given in the Appendix.

2 Nonlinear dynamical model

The dynamics of an axial dispersion reactor for one nonisothermal reaction are given by the following mass and energy balance equations:

\[
\begin{align*}
\frac{\partial x_1}{\partial t} &= D_1 \frac{\partial^2 x_1}{\partial z^2} - \nu \frac{\partial x_1}{\partial z} - k_0 x_1 + \alpha \delta(1 - x_2) \exp \left( \frac{\mu x_1}{1 + x_1} \right) \\
\frac{\partial x_2}{\partial t} &= D_2 \frac{\partial^2 x_2}{\partial z^2} - \nu \frac{\partial x_2}{\partial z} + \alpha (1 - x_2) \exp \left( \frac{\mu x_1}{1 + x_1} \right)
\end{align*}
\]

(1)

with the boundary conditions

\[
D_1 \frac{\partial x_1}{\partial z}(z = 0, t) - \nu x_1(z = 0, t) = 0; \quad i = 1, 2
\]

(2)

\[
D_1 \frac{\partial x_2}{\partial z}(z = 1, t) = 0; \quad i = 1, 2
\]

(3)

Here, $x_1 = \frac{T - T_{in}}{T_{in}}$ is the normalized temperature and $x_2 = \frac{C_{in} - C}{C_{in}}$ is the normalized concentration of the reactant. The index "in" holds for the values in the process inlet. $T_{in}$ and $C_{in}$ are constant reference values of the inlet

*Corresponding author
temperature and reactant concentration, respectively. Here, we assume that the inlet temperature is equal to the heat exchanger temperature. Observe that for all \( z, 0 \leq z \leq 1 \), and for all \( t \geq 0 \), we have: \( x_1(z, t) \geq -1 \) and \( 0 \leq x_2(z, t) \leq 1 \). (see e.g. [4]). In addition, the real constants \( D_1, D_2, \alpha, k_0, \nu \) and \( \mu \) are positive, whereas the constant \( \delta \) is positive in case of exothermic reactions and negative in case of endothermic reactions. In line with [9], we consider the Hilbert space \( H = L^2[0, 1] \times L^2[0, 1] \) with the usual inner product. If we define \( x(t) = (x_1(t), x_2(t))^T \), the state-space description is given by the following (abstract) differential equation on the Hilbert space \( H \):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + N(x(t)), \\
x(0) &= x_0 \in H
\end{align*}
\]

where \( A \) is the linear (unbounded) operator defined on its domain \( \mathcal{D}(A) = \{ x = (x_1, x_2)^T \in H \mid x \text{ and } \frac{dx}{dz} \in H \} \) are absolutely continuous \( \frac{d^2x}{dz^2} \in H \) and

\[
D_1 \frac{dx_i}{dz}(0) - \nu x_i(0) = D_1 \frac{dx_i}{dz}(1) = 0; \quad i = 1, 2
\]

by

\[
A x = \begin{pmatrix}
D_1 \frac{\partial^2 x_1}{\partial z^2} - \frac{\partial x_1}{\partial z} - k_0 x_1 & 0 \\
0 & D_2 \frac{\partial^2 x_2}{\partial z^2} - \nu \frac{\partial x_2}{\partial z}
\end{pmatrix}
\]

and the nonlinear operator \( N \) is defined on

\[
D := \{ (x_1, x_2)^T \in H \mid 0 \leq x_2(z) \leq 1 \text{ and } x_1(z) \geq -1, \text{ for almost all } z \in [0, 1] \}
\]

by

\[
N(x) = \begin{pmatrix}
\alpha \delta(1 - x_2) \exp \left( \frac{\mu x_1}{1 + x_1} \right), \\
\alpha (1 - x_2) \exp \left( \frac{\mu x_1}{1 + x_1} \right)
\end{pmatrix}
\]

As in [9], it can be shown that the operators \( A_1 \) and \( A_2 \) are the infinitesimal generators of exponentially stable \( C_0 \)-semigroups \( (T_1(t))_{t \geq 0} \) and \( (T_2(t))_{t \geq 0} \); whence by using standard arguments (e.g. [2, Lemma 3.2.2]), the linear operator given by (5)-(6) is the infinitesimal generator of the exponentially stable \( C_0 \)-semigroup of bounded linear operators on \( H \) given by

\[
T(t) = \begin{pmatrix}
T_1(t) & 0 \\
0 & T_2(t)
\end{pmatrix}
\]

Moreover, as the nonlinear operator \( N \) is locally Lipschitz continuous, equation (4) has a unique mild solution on some interval \([0, t_{\max})\), \((t_{\max} \in \mathbb{R})\) given by (see e.g. [7, p. 185-186]):

\[
x(t) = T(t)x_0 + \int_0^t T(t-s)N(x(s))ds, \quad 0 \leq t < t_{\max}
\]

(10)

Now, in order to investigate the asymptotic behavior of the state trajectories, we need the existence of solutions on the whole interval \([0, +\infty)\) (i.e. with \( t_{\max} = +\infty \)). This existence will be proved by using some tools from nonlinear functional analysis, which are recalled in the next section.

3 An existence and uniqueness preliminary result

Let \( X \) be a Hilbert space and \((T(t))_{t \geq 0}\) a \( C_0 \)-semigroup of linear operators such that \( \| T(t) \| \leq e^{|\nu|t}, \forall t \geq 0 \), for some \( \nu \in \mathbb{R} \). Let \( A \) be the infinitesimal generator of \((T(t))_{t \geq 0}\) and \( D \) be a closed subset of \( X \). Assume also that \( N \) is a continuous function from \( D \) into \( X \). We consider the following initial value problem:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + N(x(t)), \\
x(0) &= x_0 \in D
\end{align*}
\]

(11)

Let \( I \) denote the identity operator on \( X \). For \( y \in X \), define the distance from \( y \) to \( D \) by

\[
d(y; D) = \inf_{x \in D} d(y, x)
\]

(12)

where \( d(y, x) \) denotes the distance induced by the norm of the Hilbert space \( X \). The following result, proved in [5, p. 355], gives sufficient conditions for the existence and the uniqueness of the mild solution of system (11) on the whole interval \([0, +\infty)\).

Theorem 3.1 Assume that

i) \( D \) is \( T(t) \)-invariant; i.e. \( T(t)D \subset D \), for all \( t \geq 0 \);

ii) for all \( x \in D \),

\[
\lim_{h \to 0^+} \frac{1}{h} d(x + hN(x); D) = 0 ;
\]

iii) \( N \) is continuous on \( D \) and there exists \( l_N \in \mathbb{R}^+ \) such that the operator \((N - l_N I)\) is dissipative on \( D \).

Then, equation (11) has a unique mild solution \( x(t, x_0) \) on \([0, +\infty)\), for all \( x_0 \in D \). Moreover, if \((S(t))_{t \geq 0}\) is defined on \( D \) by \( S(t)x_0 = x(t, x_0) \), for all \( t \geq 0 \) and \( x_0 \in D \), it is a nonlinear semigroup on \( D \), with \((A + N)\) as its generator.

4 Preliminary lemmas

Let us first recall some definitions related to the concept of positive semigroups. We refer the reader to [6] and [8] for more details.

Let \( E \) be a real Banach lattice and \( E^+ \) be the positive cone
that introduces in $E$ a partial order relation defined by: for all $x, y \in E$,

$$x \geq y \text{ if and only if } x - y \in E^+$$

We have therefore $E^+ := \{ x \in E : x \geq 0 \}$. Let $\Gamma$ be a linear operator on $E$, then $\Gamma$ is said a positive operator if

$$\Gamma x \geq 0 \text{ for all } x \geq 0$$

or equivalently $\Gamma E^+ \subset E^+$. And we can introduce the following definition:

**Definition 4.1** A family of bounded linear operators $\{\Gamma(t), t \geq 0\}$ of $E$ is said to be a positive $C_0$-semigroup on $E$ if $\{\Gamma(t), t \geq 0\}$ is a $C_0$-semigroup on $E$ and $\Gamma(t)$ is a positive operator for all $t \geq 0$.

The rest of the section is devoted to the proof of technical results that will be useful to establish the positivity of the semigroups $T_1(t)$ and $T_2(t)$ defined on $L^2[0, 1]$, by their generators $A_1$ and $A_2$, respectively (see (5) and (6)). Note that $L^2[0, 1]$ is a real Banach lattice whose positive cone is given by:

$$L^2[0, 1]^+ := \{ h \in L^2[0, 1] : h \geq 0 \text{ a.e.} \}$$

Let us consider the differential operator $B$ on $L^2[0, 1]$ given by:

$$Bh = \frac{1}{w}(-\frac{d}{dz}(p \frac{dh}{dz}) + qh) \quad (13)$$

where $w(z), p(z), \frac{dp}{dz}(z), q(z)$ are real continuous functions on the interval $[0, 1]$ and $p(z) > 0, w(z) > 0$. The domain of $B$ is:

$$D(B) = \left\{ h \in L^2[0, 1], h, \frac{dh}{dz} \in L^2[0, 1] \text{ are absolutely continuous, } \frac{d^2h}{dz^2} \in L^2[0, 1] \text{ and } P_1h = \Delta_1 \frac{dh}{dz}(0) - v_1h(0) = 0; \right. \quad P_2h = \Delta_2 \frac{dh}{dz}(1) - v_2h(1) = 0 \right\} \quad (14)$$

where we assume that $\Delta_1, \Delta_2, v_1, v_2$ are real constants verifying $|\Delta_1| + |v_1| > 0$ and $|\Delta_2| + |v_2| > 0$. Therefore, if $0$ is not in the spectrum of $B$ then: (see [2], p82)

$$(B^{-1}h)(x) = \int_0^1 g(x, y)h(y)w(y)dy \quad (15)$$

where:

$$W(0)p(0)g(x, y) = \begin{cases} -h_1(x)h_2(y), & 0 \leq x \leq y \leq 1 \\ -h_2(x)h_1(y), & 0 \leq y \leq x \leq 1 \end{cases} \quad (16)$$

and $h_1$ and $h_2$ are linearely independent solutions of:

$$p \frac{d^2h_1}{dx^2} + \frac{dp}{dx}\frac{dh_1}{dx} - qh_1 = 0, \quad P_1h_1 = 0, P_2h_1 \neq 0 \quad (17)$$

$$p \frac{d^2h_2}{dx^2} + \frac{dp}{dx}\frac{dh_2}{dx} - qh_2 = 0, \quad P_1h_2 \neq 0, P_2h_2 = 0 \quad (18)$$

$$W(0) = h_1(0)\frac{dh_2}{dx}(0) - h_2(0)\frac{dh_1}{dx}(0) \quad (19)$$

Let us now consider the particular case where, for some $\Delta > 0$ and $\lambda > 0$, $p(z) = \Delta \exp(-\frac{\lambda}{z})$, $w(z) = \exp(-\frac{\lambda}{z})$, and $q(z) = \lambda \exp(-\frac{\lambda}{z})$. Then the operator $B$ given by (13) becomes:

$$Bh = (\lambda I - \tilde{A})h \quad (20)$$

where $\tilde{A}$ is defined on $D(B)$ by:

$$\tilde{A}h = \Delta \frac{\partial^2h}{\partial z^2} - \nu \frac{\partial h}{\partial z} \quad (21)$$

It is shown in [9] that the operator $\tilde{A}$ is the generator of an exponentially stable $C_0$-semigroup on $L^2[0, 1]$, which is denoted by $\Lambda(t)$.

**Lemma 4.1** The semigroup $\Lambda(t)$ is positive.

**Proof:** We shall show that the resolvent operator of $\tilde{A}$, $R(\lambda, \tilde{A})$, is positive for all $\lambda > 0$. Because of the exponential stability of $\Lambda(t)$, $0$ is not in the spectrum of $B$ for all $\lambda > 0$. Therefore:

$$B^{-1}h = R(\lambda, \tilde{A})h \quad (22)$$

So : $R(\lambda, \tilde{A})$ is given by (15).

Let us choose:

$$h_1(z) = \exp(\nu r_1 z) - \frac{r_2}{r_1} \exp(\nu r_2 z) > 0 \quad (23)$$

$$h_2(z) = -\frac{r_2}{r_1} \exp(\nu r_2 - \nu r_1) \exp(\nu r_1 z) + \exp(\nu r_2 z) > 0 \quad (24)$$

With:

$$r_1 = \frac{v - \sqrt{v^2 + 4\Delta\lambda}}{2\Delta} < 0 \quad (25)$$

$$r_2 = \frac{v + \sqrt{v^2 + 4\Delta\lambda}}{2\Delta} > 0 \quad (26)$$

Therefore, we have $h_1 > 0$ and $h_2 > 0$. Let us now compute $W(0)$:
W(0) = (1 - \frac{r_2}{r_1})(-r_2 \exp(r_2 - r_1) + r_2)
-(-\frac{r_2}{r_1} \exp(r_2 - r_1) + 1)(r_1 - \frac{r_2}{r_1})
= (1 - \frac{r_2}{r_1})[-r_2 \exp(r_2 - r_1) + r_2]
+ r_2(1 + \frac{r_2}{r_1})(\frac{r_2}{r_1} \exp(r_2 - r_1) - 1)]
= (1 - \frac{r_2}{r_1})[-r_2 \exp(r_2 - r_1) + r_2 - (r_1 + r_2)
+ \frac{r_2}{r_1} \exp(r_2 - r_1) + r_2 \exp(r_2 - r_1)](1 - \frac{r_2}{r_1})
\leq r_2(1 - \frac{r_2}{r_1})[-1 + \frac{r_2}{r_1} \exp(r_2 - r_1)]
\leq 0
(27)\quad (28)

Now by (16) we can conclude that \(g(x, y) \geq 0\), which implies by using (15) that:
\[R(\lambda, \tilde{A})h \geq 0\] for all \(h \geq 0\), for all \(\lambda > 0\).

Finally, by using the exponential formula [7]:
\[\Lambda(t)h = \lim_{n \to \infty} \left[1 - R(\frac{t}{n}, \tilde{A})\right]^{nh}\]
we have for all \(t > 0\)
\[\Lambda(t)h \geq 0\]
for all \(h \geq 0\). This completes the proof.

\textbf{Lemma 4.2}
\[\lambda R(\lambda, \tilde{A}) \leq I \quad \text{for all} \quad \lambda > 0\]
where \(I\) is the function identically equal to 1.

\textbf{Proof:} We have by (22) and (15), for all \(z \in [0, 1]\):
\[\{R(\lambda, \tilde{A})I\}(z) = \int_0^1 g(z, y)I(y)w(y)dy
= \int_0^z g(z, y)(y)w(y)dy
+ \int_z^1 g(z, y)(y)w(y)dy\]
From (16) one has:
\[-p(0)W(0)[R(\lambda, \tilde{A})I](z) =
\quad h_2(z) \int_0^z h_1(y)w(y)dy + h_1(z) \int_z^1 h_2(y)w(y)dy\]
(29)

From the computation of the integrals done in the Appendix (see (39) and (40), we get
\[-p(0)W(0)[R(\lambda, \tilde{A})I](z) =
-\frac{r_2}{r_1}(\frac{1}{r_2} - \frac{r_2}{r_1}) \exp(r_2 - r_1) \exp(r_1z) + \left(\frac{1}{r_2} - \frac{r_2}{r_1}\right) \exp(r_2z)
-\left(\frac{r_2}{r_1} - \frac{r_2}{r_1}\right) \exp(r_2 - r_1) + \frac{1}{r_1} - \frac{1}{r_2}\]
(30)

But the function \(z \in [0, 1] \to F(z) = -p(0)W(0)[R(\lambda, A)I](z)\) is nondecreasing (as shown in the Appendix). Therefore:
\[-p(0)W(0)[R(\lambda, \tilde{A})I](z) \leq -p(0)W(0)[R(\lambda, \tilde{A})I](1)\]
(31)
So by (30) we can write that:
\[-p(0)W(0)[R(\lambda, \tilde{A})I](1) =
\left(-\frac{1}{r_1} + \frac{r_2}{r_1} + \frac{r_2}{r_1} - \frac{r_2}{r_1}\right) \exp(r_2)
+ \frac{1}{r_1} - \frac{1}{r_2} + \frac{r_2}{r_1} - \frac{r_2}{r_1} \exp(r_2 - r_1)\]
(32)
And by the fact that \(r_1 < 0, r_2 > 0 \) and \(|r_1| < r_2\), we get for all \(z \in [0, 1]\):
\[\left(-\frac{1}{r_1} + \frac{r_2}{r_1} + \frac{1}{r_2} - \frac{r_2}{r_1}\right) < 0\]
which, combined with (32), gives:
\[-p(0)W(0)[R(\lambda, \tilde{A})I](1) \leq
\frac{1}{r_1} - \frac{1}{r_2} + \frac{r_2}{r_1} - \frac{r_2}{r_1} \exp(r_2 - r_1)\]
(33)
Therefore, by (31) and (33), we have for all \(z \in [0, 1]\):
\[-p(0)W(0)[R(\lambda, \tilde{A})I](z) \leq \frac{1}{r_1} - \frac{1}{r_2} + \frac{r_2}{r_1} - \frac{r_2}{r_1} \exp(r_2 - r_1)\]
\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{r_2}{r_1} - \frac{r_2}{r_1} \exp(r_2 - r_1)\right] \leq \frac{1}{r_1r_2}(-W(0))\]
Where the last inequality is deduced from (27). As \(W(0) < 0\) and \(p(0) = \Delta > 0\), we get for all \(z \in [0, 1]\):
\[\{R(\lambda, \tilde{A})I\}(z) \leq \frac{-1}{r_1r_2\Delta} \leq \frac{1}{\lambda}\]
whence \(\lambda R(\lambda, \tilde{A}) \leq I\). This completes the proof.
5 Main results

Recall that we are interested in the following initial value problem:

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + N(x(t)) \\
x(0) &= x_0 \in D
\end{aligned}
\]  

(34)

where \( A \) and \( N \) are given by (4)-(8) and \( D \) is the closed subset of \( H \) defined by:

\[
D = \{(x_1, x_2)^T \in H \text{ such that } 0 \leq x_2(z) \leq 1 \\
\text{and } x_1(z) \geq -1, \text{ for almost all } z \in [0, 1]\}
\]  

(35)

Clearly, \( D \) is a closed convex subset of \( H \) and the nonlinear operator \( N \), given by (8), is well defined and continuous on \( D \). Actually, in order to be able to apply Theorem 3.1, we need to prove that \( D \) is \( T(t) \)-invariant. This can be done using the following two technical lemmas.

**Lemma 5.1** The \( C_0 \)-semigroup \( T(t) \) given by (9) is positive.

**Proof:** Note that the positivity of the \( C_0 \)-semigroup \( T(t) \) is equivalent to the positivity of the \( C_0 \)-semigroups \( T_1(t) \) and \( T_2(t) \). So if we take, in (21), \( \Delta = D_2, v = \nu, \Delta_1 = \Delta_2 = D_2 \) and \( v_1 = v_2 = \nu \) in (14) then \( \hat{A} = A_2 \). So the \( C_0 \)-semigroup \( \Lambda(t) = T_2(t) \) is positive by Lemma 4.1.

In order to prove the positivity of \( T_1(t) \), let us take \( \Delta = D_1 \) and \( v = \nu \) in (21) and \( \Delta_1 = \Delta_2 = D_1 \) and \( v_1 = v_2 = \nu \) in (14). Therefore, \( A_1 = \lambda \) and \( T_1(t) = \exp(-k_0t)\Lambda(t) \).

Thus, the positivity of \( \Lambda(t) \) implies the positivity of \( T_1(t) \), by using Lemma 4.1.

**Lemma 5.2**

For \( i = 1, 2 \), \( \lambda R(\lambda, A_i) I \leq I \) for all \( \lambda > 0 \) (36)

where \( I \) is the function identically equal to 1.

**Proof:** As in the proof of Lemma 5.1, we proceed in two steps.

First, for \( \hat{A} = A_2 \), we have for all positive \( \lambda \),

\[
\lambda R(\lambda, A_2) I = \lambda R(\lambda, \hat{A}) I \\
\leq I
\]

where the last inequality is given by Lemma 4.2.

Secondly, for \( A_1 = \hat{A} - k_0 I \), using the positivity of the resolvent operator \( R(\lambda, \hat{A}) \) and Lemma 4.2, we get:

\[
\lambda R(\lambda, A_1) I = \lambda R(\lambda + k_0, \hat{A}) I \\
\leq (\lambda + k_0)R(\lambda + k_0, \hat{A}) I \\
\leq I
\]

where the positivity of \( R(\lambda, \tilde{A}) \) is deduced from the positivity of the corresponding semigroup. This completes the proof.

We are now able to state the following lemma.

**Lemma 5.3**

\( T(t)D \subset D \) for all \( t \geq 0 \)

where \( D \) is given by (7).

**Proof:** Let \((x, y) \in D\). We can then write in matrix representation:

\( (-I, 0)^T \leq (x, y)^T \leq (x, I)^T \)

Hence, by the positivity of \( T(t) \), we have

\( (-T_1(t)I, 0)^T \leq T(t)(x, y)^T \leq (T_1(t)x, T_2(t)I)^T \)

whence the invariance of the set \( D \) holds if the following inequalities hold for all \( t \geq 0 \):

\[
\begin{aligned}
- I \leq -T_1(t) I & \quad (37) \\
T_2(t) I \leq I & \quad (38)
\end{aligned}
\]

Now \( \lambda R(\lambda, A_1) I \leq I \) and \( \lambda R(\lambda, A_2) I \leq I \) for all \( \lambda > 0 \) (see Lemma 5.2). Then, by using the exponential formula we get (37) and (38).

We are now ready to state our main result.

**Theorem 5.1** For every \( x_0 \in D \), equation (4) has a unique mild solution \( x(t, x_0) \) on the interval \([0, +\infty[. Moreover, if we set \( S(t)x := x(t, x_0) \), then \( (S(t))_{t \geq 0} \) is a strongly continuous nonlinear semigroup on \( D \), generated by the operator \( A + N \).

**Proof:** By using the same arguments as those used in [9], one can show that the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) given by (9) is exponentially stable, i.e. there exists \( w < 0 \) and \( M \geq 1 \) such that \( \| T(t) \| \leq Me^{wt}, \) for all \( t \geq 0 \). Now, by endowing \( H \) with the norm:

\[
\| x \| := \sup \{ \exp(-wt) \| T(t)x \|, t \geq 0 \},
\]

which is equivalent to the norm \( \| \cdot \| \) (see e.g. [5, p. 277] or [7, p. 19]), it follows that \( \| T(t) \| \leq e^{wt} \).

In addition, in order to establish that there exists \( h \in R^+ \) such that the operator \( (N - l_N I) \) is dissipative on \( D \) with respect to \( \| \cdot \| \), it suffices to prove that \( N \) is a Lipschitz operator on \( D \) with respect to \( \| \cdot \| \) (5, Lemma 6.1, p. 245)).

Actually \( N \) is a Lipschitz operator relatively to the norm \( \| \cdot \| \) with Lipschitz constant denoted here by \( l_1 \), as proved in ([11, Lemma 4.2 ). Hence, \( N \) is also a Lipschitz operator relatively to the equivalent norm \( \| \cdot \| \) with Lipschitz constant \( l_2 = l_1 M \). Consequently \( (N - l_N I) \) is a dissipative operator on \( D \), which is the third condition of theorem 3.1.

Now the first condition of Theorem 3.1 is stated in Lemma 5.3. Finally, the proof of the fact that the second condition of Theorem 3.1 is satisfied can be done as in lemma 4.1 of [11].

So all the assumptions of Theorem 3.1 hold and the result is proved.
6 Concluding remark

The analysis performed here is a preliminary step of a study dedicated to the design of controllers for axial dispersion nonisothermal reactors based on the nonlinear PDE model of the process. In this paper we have studied the existence and the uniqueness of the state (temperature and reactant concentration) trajectories of an axial dispersion nonisothermal reactor model by using infinite dimensional system techniques. An important open question is the existence, multiplicity and stability of equilibrium points for this model, under physically meaningful assumptions.

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Appendix

1. Computation of the integrals in (29):

In view of (16), there holds:

\[-p(0)W(0)[R(\lambda, \bar{A}) I](z) = h_2(z) \int_0^1 h_1(y) w(y) dy + h_1(z) \int_1^2 h_2(y) w(y) dy\]

Hence, using (23) and (24):

\[h_2(z) \int_0^z h_1(y) \exp\left(-\frac{y}{\Delta}\right) dy = h_2(z) \int_0^z \left[\exp(r_1 y) - \frac{r_2}{r_1} \exp(r_2 y)\right] \exp\left(-\frac{y}{\Delta}\right) dy\]

\[= h_2(z) \left[\exp(r_1 y) - \frac{r_2}{r_1} \exp(r_2 y)\right] \int_0^y dy\]

\[= h_2(z) \left[\exp(r_1 y) - \frac{r_2}{r_1} \exp(r_2 y)\right] \int_0^y dy\]

\[= h_2(z) \left[\frac{1}{r_1} \exp(r_1 y) - \frac{r_2}{r_1} \exp(r_2 y)\right] + \frac{1}{r_2} - \frac{r_2}{r_1}\]
is decreasing on \([0, 1]\). Therefore: \(f(z) > f(0)\) and \(g(z) > g(1)\) for all \(z \in [0, 1]\). So we have:

\[
\frac{dF}{dz}(z) > f(0) + g(1)
\]

\[
= -(1 - \frac{r_2^2}{r_1^2}) \exp(r_2 - r_1) + (1 - \frac{r_2^2}{r_1^2}) \exp(r_2)
\]

\[
= \exp(r_2)(1 - \frac{r_2^2}{r_1^2})[1 - \exp(-r_1)]
\]

But \(r_1 < 0\) and \((1 - \frac{r_2^2}{r_1^2}) < 0\) so \(\frac{dF}{dz}(z) > 0\) for all \(z \in [0, 1]\) which completes the proof.

References


