Sensors and actuators for compensation in hyperbolic systems

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Abstract

We introduce and characterize in the case of hyperbolic systems, the notion of remediability, which consists to spatially compensate, in finite time, a disturbance by a convenient choice of actuators. Such actuators are said to be efficient. We study the relationship between these different notions and controllability as well as strategic actuators. We also determine the set of remediable disturbances, and we give the optimal control which compensates the disturbance of the system. Finally, an application is given in one and two dimensional cases.

1 Introduction

Disturbances problems have been studied in previous works [9,10]. Another approach, based on actuators and sensors notions, has been developed in [3,4,5] for diffusion systems. This work is an extension of this approach to a class of hyperbolic systems.

We consider, without loss of generality, the system described by the following hyperbolic equation

\[
\begin{aligned}
\frac{\partial^2 x}{\partial t^2} (\xi, t) &= \Delta x(\xi, t) + Bu(t) + f(\xi, t) \\
x(\xi, 0) &= \frac{\partial x}{\partial t} (\xi, 0) = 0 \text{ on } \Omega \\
x(\eta, t) &= 0 \text{ on } \partial \Omega \times [0, T]
\end{aligned}
\]

(1)

where \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \), with a sufficiently regular boundary \( \Gamma = \partial \Omega \). \( B \in \mathcal{L}(U, L^2(\Omega)) \), \( u \in L^2(0, T, U) \) with \( T > 0 \) sufficiently large; \( U \) is a Hilbert space representing the control space and \( \Delta \) is the Laplacian operator. The disturbance term \( f \in L^2(0, T; L^2(\Omega)) \) is generally unknown. The system (1) is augmented by the output equation

\[
y(t) = \begin{pmatrix} C_1 x(., t) \\ C_2 \frac{\partial x}{\partial t}(., t) \end{pmatrix}
\]

(2)

where \( C_1 \in \mathcal{L}(L^2(\Omega), Y_1) \) (and then \( C_1 \in \mathcal{L}(H^1_0(\Omega), Y_1) \)), \( C_2 \in \mathcal{L}(L^2(\Omega), Y_2) \), \( Y_1 \) and \( Y_2 \) are two Hilbert spaces (observation spaces).

Let \( A \) be the operator defined by \( A \psi = \Delta \psi \) for \( \psi \in D(A) = H^2(\Omega) \cap H^1_0(\Omega) \), and \( z = \begin{pmatrix} x \\ \frac{\partial x}{\partial t} \end{pmatrix} \in L^2(0, T; \mathcal{E}) \) where \( \mathcal{E} = H^1_0(\Omega) \times L^2(\Omega) \). The system (1) is equivalent to

\[
\begin{aligned}
\dot{z}(t) &= Az(t) + Bu(t) + \tilde{f}(t) \\
z(0) &= 0
\end{aligned}
\]

(3)

and the output equation can be written

\[
y(t) = Cz(t)
\]

(4)

where the operator \( A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \), with \( D(A) = D(A) \times H^1_0(\Omega) \). The adjoint \( A^* \) of \( A \) is given by \( A^* = -A \).

The operator \( B \) is defined by \( B = \begin{pmatrix} 0 \\ B \end{pmatrix} \) and its adjoint is defined by \( B^* = (0 \ B^*) \). \( \tilde{f} = \begin{pmatrix} 0 \\ f \end{pmatrix} \) and \( C \in \mathcal{L} (\mathcal{E}, Y) \) is defined by \( C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \), with \( Y = Y_1 \times Y_2 \).

The operator \( A \) is linear, closed with a dense domain in the state space \( \mathcal{E} \), and generates on \( \mathcal{E} \) a strongly continuous semi-
where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$ and $(\varphi_{nj})_{j=1,\ldots,n_{r_n}}$ is an orthonormal basis of eigenvectors of $A$, associated to eigenvalues $\lambda_n < 0$ with a multiplicity $r_n$. The adjoint semi-group is defined by $S^\ast(t) = S(-t) \quad \forall t \geq 0$. $\mathcal{E}$ is a Hilbert space for the inner product $\langle z, z' \rangle_\mathcal{E} = \langle -(A)^2 z_1, -(A)^2 z'_1 \rangle + \langle z_2, z'_2 \rangle$ for $z = (z_1, z_2)$ and $z' = (z'_1, z'_2) \in \mathcal{E}$.

If $f = 0$, $u = 0$, the observation, noted $y_{u,f}$, is normal and it’s null. But if $f \neq 0$, the observation is disturbed and the problem is to find an input operator $B$ (actuators), with respect to the output operators $C_1$ et $C_2$ (sensors), ensuring the finite time compensation of any disturbance, i.e. for any $f \in L^2(0,T;L^2(\Omega))$, there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$\int_0^T \mathcal{C}S(t-s)F(s)ds + \int_0^T \mathcal{C}S(T-s)Bu(s)ds = 0$$

or for any $f \in L^2(0,T;L^2(\Omega))$ and any $\epsilon > 0$, there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$\| \int_0^T \mathcal{C}S(t-s)F(s)ds + \int_0^T \mathcal{C}S(T-s)Bu(s)ds \| < \epsilon$$

This is the basic notion of remediability, which is, as it will be shown, weaker than the controllability.

This paper is organized as follows: In section 2, we briefly recall in the case of the considered system, the notions of controllability, observability, strategic actuators and sensors, which will be used later in this work. In section 3, we define and characterize the notion of exact and weak remediability, as well as efficient actuators. We study in section 4, the relationship between controllability and remediability, and hence between strategic actuators and efficient actuators. In section 5, we study the problem of exact remediability with minimum energy (cost) using an extention of Hilbert Uniqueness Method (H.U.M.). Then, we characterize the set of exactly remediable disturbances, and we give the optimal control which compensate spatially an arbitrary disturbance. Finally, we give an application in one and two space dimensions.

2 Preliminaries

2.1 Controllability

We consider the system

$$\begin{cases}
\dot{z}(t) = Az(t) + Bu(t) & 0 < t < T \\
z(0) = 0
\end{cases} \quad (5)$$

The solution of system (5), is noted $z_u(.)$. At final time $T$, we have

$$z_u(T) = Hu = \left( \begin{array}{c}
\sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_0^T <Bu(s), \varphi_{nj}> \\
\sum_{n \geq 1} \sum_{j=1}^{r_n} \int_0^T cos(\sqrt{-\lambda_n}(T-s))ds \varphi_{nj}
\end{array} \right)$$

$\mathcal{E}$. For $u \in L^2(0,T;\mathcal{U})$. The adjoint operator of $H$ is defined by

$$H^* = B^* \mathcal{S}^\ast(T-.).$$

2.1.1 Definition and characterization

Definition 2.1

The system (5) is said to be

i) exactly controllable on $[0,T]$ if for every $z_d \in \mathcal{E}$, there exists $u \in L^2(0,T;\mathcal{U})$ such that $z_u(T) = z_d$.

ii) weakly controllable on $[0,T]$ if for every $z_d \in \mathcal{E}$, and every $\epsilon > 0$, there exists $u \in L^2(0,T;\mathcal{U})$ such that

$$\| z_u(T) - z_d \| < \epsilon$$

where $X = L^2(\Omega) \times L^2(\Omega)$ Let $\mathcal{E}', \mathcal{U}'$ be the dual spaces of $\mathcal{E}$ and $\mathcal{U}$ respectively, then we have the following result [6,7].

Proposition 2.2

The system (5) is

i) exactly controllable on $[0,T]$ if

$$\Leftrightarrow \text{Im}H = E$$

$$\exists \gamma > 0 \text{ such that, } \forall z^* \in \mathcal{E}' \quad \| z^* \|_{\mathcal{E}'} \leq \gamma \| B^* \mathcal{S}^\ast(T-.).z^* \|_{L^2(0,T;\mathcal{U}')}$$

ii) weakly controllable on $[0,T]$ if

$$\Leftrightarrow \text{Im}H = E$$

$$\Leftrightarrow \ker(H^*) = \{ 0 \}$$

The exact controllability implies the weak controllability, the converse is not true [6,7].

2.1.2 Controllability and actuators

In the case of $p$ actuators $(\Omega_i, g_i)_{i=1,p}$ [7], we have $\mathcal{U} = \mathbb{R}^p$ and

$$B : \mathbb{R}^p \longrightarrow \mathcal{E}$$

$$u(t) \longrightarrow B(u(t)) = (0, \sum_{i=1}^p g_i u_i(t))^{tr}$$
where \( u = (u_1, ..., u_p)^{tr} \in L^2(0, T; \mathbb{R}^p) \) and \( g_i \in L^2(\Omega_i) \), \( \Omega_i = \text{supp}(g_i) \subset \Omega \) for \( i = 1, p \) and \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \), we have
\[
\mathcal{B}^* z = (\langle g_1, z_2 \rangle_{\Omega_i}, ..., \langle g_p, z_2 \rangle_{\Omega_j})^{tr}
\]
for \( z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \in \mathcal{E}', \) where generally \( M^{tr} \) denotes the transposal matrix of \( M \).

**Definition 2.3**

We say that the actuators \((\Omega_i, g_i)_{i=1,p}\) are strategic, if the corresponding system (5) is weakly controllable in finite time.

We have the following characterisation result [6,7].

**Proposition 2.4**

The actuators \((\Omega_i, g_i)_{i=1,p}\) are strategic, if and only if
\[
\begin{cases} 
  i) & p \geq r_n , \quad \forall \ n \geq 1 \\
  ii) & \text{rank}(M_n) = r_n , \quad \forall \ n \geq 1 \\
  \text{where} & M_n = (\langle g_i, \varphi_{nj} \rangle_{\Omega_j})_{i=1,p} \end{cases}
\]

The characterization of strategic pointwise actuators is similar to (6), with \( M_n = (\varphi_{nj}(b_i))_{i=1,p} \), where \( b_i \) are the actuators locations.

3 Remediability

3.1 Considered system and notations

We consider the system (3) augmented by (4). The solution of (3), noted \( z_{u,f} \), is given by
\[
z_{u,f}(t) = \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds
\]
Let \( \hat{H} \) be the linear operator defined by
\[
\hat{H} : L^2(0, T; \{0\} \times L^2(\Omega)) \longrightarrow \mathcal{E}
\]
\[
\hat{f} \mapsto \hat{H}\hat{f} = \int_0^T S(T-s)f(s)ds
\]
we have
\[
z_{u,f}(T) = Hu + \hat{H}\hat{f}
\]
Let \( R \) be the linear operator defined by
\[
R : L^2(0, T; \{0\} \times L^2(\Omega)) \longrightarrow Y
\]
\[
\hat{f} \mapsto R\hat{f} = CH\hat{f}
\]
The adjoint operator of \( R \) is given by
\[
R^* : Y' \longrightarrow L^2(0, T; \{0\} \times L^2(\Omega))
\]
\[
\theta \mapsto R^* \theta = \hat{H}^*C^*\theta = S^*(T - .)C^*\theta
\]
We have then
\[
y_{u,f}(T) = CHu + RF\hat{f}
\]

3.2 Remediability

In the "normal" case, i.e. without disturbance and control, the system (3) is given by
\[
\begin{cases} 
  \dot{z}(t) = Az(t) ; \quad 0 < t < T \\
  z(0) = 0 
\end{cases}
\]
the state \( z(.) \) is given by \( z(t) = 0 \), then the observation is \( y(t) = 0 \). But if the system is disturbed by \( f \) on \([0, T]\), then the observation becomes
\[
y(t) = \int_0^t CS(t-s)\hat{f}(s)ds \neq 0
\]
Then we introduce a control term \( Bu \), in order to compensate this disturbance at the final time \( T \). We have the following definitions.

**Definition 3.1**

i) We say that the system (3) augmented by the output equation (4) (or (3) + (4)) is exactly remediable on \([0, T]\), if for any \( \hat{f} \in L^2(0, T; \{0\} \times L^2(\Omega)) \), there exists \( u \in L^2(0, T; U) \) such that
\[
CHu + RF\hat{f} = 0
\]
ii) We say that (1) + (E) is weakly remediable on \([0, T]\), if for any \( \epsilon > 0 \) and \( \hat{f} \in L^2(0, T; \{0\} \times L^2(\Omega)) \), there exists \( u \in L^2(0, T; U) \) such that
\[
\| CHu + RF\hat{f} \| < \epsilon
\]
The regularity of the solution \( z_{u,f}(.) \) of the system (3) depends on that of \( f \) and on the control term \( Bu \). In the general case, we have \( z_{u,f}(.) \in L^2(0, T; V) \) where \( V \) is a Hilbert space such that \( V' \subset X \subset V \), with continuous injections.

3.3 Characterization

We have
\[
\text{Im}(CH) \subset \text{Im}(R)
\]
because for \( f = -Bu \), we have \( RF\hat{f} = y_f = -CHu \). For the characterization of the exact remediability, we have the following proposition.

**Proposition 3.2**

The following properties are equivalent

i) (3) + (4) is exactly remediable on \([0, T]\).

ii) \( \text{Im}(R) \subset \text{Im}(CH) \)

iii) \( \exists \gamma > 0 \text{ such that} \)
\[
\| S^*(T - .)C^*\theta \|_{L^2(0,T;X')} \leq \gamma \| B^*S^*(T - .)C^*\theta \|_{L^2(0,T';U')} \quad \forall \theta \in Y'
\]
Proof

(i) $\iff$ (ii) Derives from the definition.

(ii) $\iff$ (iii) Derives from

$$ \mathbb{R}^* = S^*(T - .)\mathbb{C}^* $$

(17)

and

$$ H^*\mathbb{C}^* = B^*S^*(T - .)\mathbb{C}^* $$

(18)

and the following result.

**Lemma 3.3 [7]**

Let $X$, $Y$, $Z$ be a reflexive Banach spaces and $F \in \mathcal{L}(X, Z)$, $G \in \mathcal{L}(Y, Z)$; then the following properties are equivalent

(i) $Im(F) \subset Im(G)$

(ii)

$$ \exists \gamma > 0 \text{ such that } \|F^*z^*\|_X \leq \gamma \|G^*z^*\|_Y; \forall z^* \in \mathbb{Z}^* $$

(19)

We have $H^*\mathbb{C}^* = B^*S^*(T - .)\mathbb{C}^*$, and then using (10)

$$ H^*\mathbb{C}^* = B^*R^* $$

(20)

For the weak remediability characterization, we have the following proposition.

**Proposition 3.4**

There is equivalence between

(i) $(3) + (4)$ is weakly remediable on $[0, T]$.

(ii)

$$ Im(R) \subset \overline{Im(CH)} $$

(21)

(iii)

$$ ker(B^*R^*) = ker(R^*) $$

(22)

Proof

(i) $\iff$ (ii) Derives from the definition.

(ii) $\iff$ (iii) by considering the orthogonal and using (14) and (20).

In finite dimensional case or an output given by a finite number of sensors, there is equivalence between weak and exact remediability.

Now, in the case of $p$ actuators $(\Omega_i, g_i)_{i=1,p}$, the exact remediability characterization is given by the following proposition.

**Proposition 3.5**

(3) + (4) is exactly remediable on $[0, T]$, if and only if

$$ \exists \gamma > 0 \text{ such that } \forall \theta = (\theta_1, \theta_2) \in Y' $$

$$ \sum_{n=1}^{r_n} \sum_{j=1}^{r_n} (1 - \cos(2\sqrt{-\lambda_n}T)) $$

$$ < C^*\theta_1, \varphi_{n_j} >_\Omega + T(-\lambda_n < C^*\theta_1, \varphi_{n_j} >_\Omega^2 + < C^*\theta_2, \varphi_{n_j} >_\Omega^2) $$

$$ \leq \gamma \sum_{i=1}^{p} \sum_{n=1}^{r_n} < g_i, \varphi_{n_j} >_\Omega \left( \frac{1}{2} (1 - \cos(\sqrt{-\lambda_n}T)) \right) $$

$$ < C^*\theta_1, \varphi_{n_j} >_\Omega + < C^*\theta_2, \varphi_{n_j} >_\Omega + \frac{1}{4\sqrt{-\lambda_n}} \sin(2\sqrt{-\lambda_n}T) $$

$$ (\lambda_n < C^*\theta_1, \varphi_{n_j} >_\Omega^2 + < C^*\theta_2, \varphi_{n_j} >_\Omega^2) $$

$$ + \frac{T}{4} (-\lambda_n < C^*\theta_1, \varphi_{n_j} >_\Omega^2 + < C^*\theta_2, \varphi_{n_j} >_\Omega^2) $$

(23)

Proof

The result follows from (16) and proposition 3.2 and the fact that

$$ S^*(T - s)C^* = \left( \sum_{n=1}^{r_n} \sum_{j=1}^{r_n} (\frac{1}{\sqrt{-\lambda_n}} - 1) < C^*\theta_1, \varphi_{n_j} >_\Omega \cos(\sqrt{-\lambda_n}T - s) \right) $$

$$ - \sum_{n=0}^{r_n} \sum_{j=0}^{r_n} < C^*\theta_2, \varphi_{n_j} >_\Omega \sin(\sqrt{-\lambda_n}T - s)) \varphi_{n_j} $$

$$ \sum_{n=1}^{r_n} \sum_{j=1}^{r_n} (\frac{1}{\sqrt{-\lambda_n}} - 1) < C^*\theta_1, \varphi_{n_j} >_\Omega \cos(\sqrt{-\lambda_n}T - s) $$

$$ + < C^*\theta_2, \varphi_{n_j} >_\Omega \sin(\sqrt{-\lambda_n}T - s)) $$

$$ B^*S^*(T - s)C^* = $$

$$ \left( \sum_{n=0}^{r_n} \sum_{j=0}^{r_n} < C^*\theta_1, \varphi_{n_j} >_\Omega \sin(\sqrt{-\lambda_n}T - s) \right) $$

$$ + < C^*\theta_2, \varphi_{n_j} >_\Omega \cos(\sqrt{-\lambda_n}T - s)) $$

(24)

and

$$ \|z^*\|_E = \sum_{n=1}^{r_n} < z_1, \varphi_{n_j} >_\Omega + < z_2, \varphi_{n_j} >_\Omega $$

for $z = (z_1, z_2) \in \mathcal{E}$.

Now, if the output of the system is given by $(q_1, q_2)$ sensors $(D_i, h_i)_{i=1,q_1}$ and $(D_i', k_i)_{i=1,q_2}$, we have

$$ C^*_1 \theta_1 = \sum_{i=1}^{q_1} \theta_i^1 h_i \text{ and } C^*_2 \theta_2 = \sum_{i=1}^{q_2} \theta_i^2 k_i $$

for $\theta_1 = (\theta_i^1)_{i=1,q_1} \in \mathcal{R}^{q_1}$ and $\theta_2 = (\theta_i^2)_{i=1,q_2} \in \mathcal{R}^{q_2}$. We obtain then a characterization of the exact remediability by remplacing $C^*_1 \theta_1$ and $C^*_2 \theta_2$ by these expressions in proposition 3.5.

**3.3.1 Efficient actuators**

By analogy with strategic actuators notion, we introduce the notion of efficient actuators.
Definition 3.6
Actuators ensuring in finite time, the weak remediability of the system (3) + (4), are said to be efficient actuators.

In the case of \( p \) actuators, we have the following characterization.

**Proposition 3.7**
The actuators \((\Omega_i, g_i)_{i=1,p}\) are efficient, if and only if
\[
ker(C^*) = \bigcap_{n \geq 1} ker(M_n f_n)
\]
where, for \( n \geq 1 \)
\[
f_n : Y^r \rightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}
\]
\[
f_n(\theta) = (f_{n,1}(\theta_1), f_{n,2}(\theta_2))
\]
with
\[
f_{n,i}(\theta_i) = (C^*_i \theta_1, \varphi_{n1} > \Omega, ..., C^*_i \theta_1, \varphi_{nr} > \Omega)^{tr}
\]
for \( l = 1, 2 \).

**Proof**
We have seen, in the proposition 3.4, that (3) + (4) is weakly remediably on \([0, T]\), if and only if, \(ker(B^* R^*) = ker(R^*)\). For \( \theta = (\theta_1, \theta_2) \in Y^r\), it is easy to show that \( R^* \theta = 0 \iff \)
\[
\begin{align*}
\sum_{n \geq 1} \sum_{j=1}^{r_n} & \left( C^*_i \theta_1, \varphi_{nj} > \Omega \cos \left( \sqrt{-\lambda_n}(T - .) \right) \right) \\
& - \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( C^*_i \theta_2, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0 \\
& + \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0
\end{align*}
\]
then, for all \( n \geq 1, j = 1, r_n \) and all \( s \in [0, T] \), we have
\[
\begin{align*}
& < C^*_i \theta_1, \varphi_{nj} > \Omega \cos \left( \sqrt{-\lambda_n}(T - .) \right) \\
& - \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( C^*_i \theta_2, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0 \\
& + \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0
\end{align*}
\]
henceforth, \( \forall n \geq 1 \) and \( j = 1, r_n \), we have
\[
\begin{align*}
& < C^*_i \theta_1, \varphi_{nj} > \Omega = 0 \\
& < C^*_i \theta_2, \varphi_{nj} > \Omega = 0
\end{align*}
\]
i.e. \( C^* \theta = 0 \), and consequently
\[
ker(R^*) = ker(C^*)
\]
On the other hand
\[
B^* R^* \theta = 0 \iff \forall \ i = 1, p
\]
\[
\begin{align*}
& \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \right) \\
& \sin \left( \sqrt{-\lambda_n}(T - .) \right) + \left( C^*_i \theta_2, \varphi_{nj} > \Omega \cos \left( \sqrt{-\lambda_n}(T - .) \right) \right) < g_i, \varphi_{nj} > \Omega = 0
\end{align*}
\]
and hence
\[
\begin{align*}
\sum_{n \geq 1} \sum_{j=1}^{r_n} \sin \left( \sqrt{-\lambda_n}(T - .) \right) & \quad \sum_{j=1}^{r_n} \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \cos \left( \sqrt{-\lambda_n}(T - .) \right) \varphi_{nj} = 0 \\
& + \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( C^*_i \theta_2, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0 \\
& + \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0
\end{align*}
\]
Since \( T \) is given sufficiently large, we have for all \( n \geq 1 \)
\[
\begin{align*}
& \sum_{j=1}^{r_n} \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega < g_i, \varphi_{nj} > \Omega = 0 \\
& + \sum_{j=1}^{r_n} \left( C^*_i \theta_2, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0 \\
& + \sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} C^*_i \theta_1, \varphi_{nj} > \Omega \sin \left( \sqrt{-\lambda_n}(T - .) \right) \right) \varphi_{nj} = 0
\end{align*}
\]
then
\[
M_n f_n(\theta_1) = 0 \text{ and } M_n f_n(\theta_2) = 0 \forall n \geq 1
\]
and consequently
\[
ker(B^* R^*) = \bigcap_{n \geq 1} ker(M_n f_n)
\]
then, we have the result.

If the output is given by \((q_1, q_2)\) sensors \((D_i, h_i)_{i=1,q_1}\) and \((D_j, k_i)_{i=1,q_2}\), the characterization of efficient actuators, is given by the following proposition.

**Proposition 3.8**
The actuators \((\Omega_i, g_i)_{i=1,p}\), are efficient, if and only if
\[
\bigcap_{n \geq 1} ker(M_n G_{n,1}) \times ker(M_n G_{n,2}) = \{0\}
\]
where
\[
G_{n,1} = (h_i, \varphi_{nj} > \Omega)_{i=1,q_1} \quad j=1,r_n
\]
and
\[
G_{n,2} = (k_i, \varphi_{nj} > \Omega)_{i=1,q_2} \quad j=1,r_n
\]
**Proof**
For \( \theta = (\theta_1, ..., \theta_l)^{tr} \in \mathbb{R}^{lq} \), with \( l = 1, 2 \), we have
\[
C^*_1 \theta_1 = \sum_{i=1}^{q_1} \theta_1^i h_i \quad \text{and} \quad C^*_2 \theta_2 = \sum_{i=1}^{q_2} \theta_2^i k_i
\]
Since the functions \((h_i)_{i=1,q_1}\) (respectively \((k_i)_{i=1,q_2}\)) are linearly independent because \(D_i \cap D_j = \emptyset\) (respectively \(D_i' \cap D_j' = \emptyset\)) for \( i \neq j \), then, \(ker(C^*_l) = \{0\}\) for \( l = 1, 2 \), and using (26), we have
\[
ker(R^*) = \{0\}
\]
On the other hand, we show by the same that,\
\[ B^* R^* \theta = 0 \iff \forall \ l=1, p \ : \ \forall \ n \geq 1 \]
\[
\begin{aligned}
\sum_{j=1}^{r_n} \sqrt{-\lambda_n} \ < g_l, \varphi_{n_j} > \sum_{i=1}^{q_1} \theta_i^1 < h_i, \varphi_{n_j} > = 0 \\
\sum_{j=1}^{r_n} \sqrt{-\lambda_n} \ < g_l, \varphi_{n_j} > \sum_{i=1}^{q_2} \theta_i^2 < h_i, \varphi_{n_j} > = 0 \\
\iff M_n G^r_{n,i} \theta_i = 0, \ i = 1, 2, \ \forall \ n \geq 1, \ and \ then \\
\ker(B^* R^*) = \bigcap_{n \geq 1} \ker(M_n G^r_{n,1}) \times \ker(M_n G^r_{n,2}) \\
\end{aligned}
\]

using (22), we have the result. \( \square \)

We deduce the following corollaries.

**Corollary 3.9**

If there exists \( n_0 \) such that\n
\[ \text{rank}(M_{n_0} G^r_{n_0,1}) = q_1 \text{ and } \text{rank}(M_{n_0} G^r_{n_0,2}) = q_2 \]

then the actuators \( (\Omega_i, g_i)_{i=1,p} \) are efficient.

**Proof**

Derives from (28) and the fact that \( \ker(M_{n_0} G^r_{n_0,i}) = \{0\} \) for \( l = 1, 2 \) by using (30). \( \square \)

Let us remark that, the condition \( p \geq \sup r_n \) is necessary for actuators \( (D_i, g_i)_{i=1,p} \) to be strategic, but it is not necessary to them in order to be efficient, and the condition \( q \leq p \) is not necessary for actuators to be efficient.

On the other hand, in the case of pointwise actuators, the characterization of the remediability is given by analogous properties to those seen for zone actuators.

### 4 Remediability and controllability

In this part, we study the relationship between controllability and remediability, and therefore between strategic and efficient actuators.

**Proposition 4.1**

If (5) is exactly controllable on \([0, T]\), then (3) + (4) is exactly remediable on \([0, T]\).

**Proof**

For \( \theta \in Y^* \), we have
\[
\| S^*(T - .) C^* \theta \|_{L^2(0,T; E')}^2 = \int_0^T \| S^*(T - s) C^* \theta \|_{E'}^2 \ ds \\
\leq \int_0^T \| S^*(T - s) \|_{L^2(0,T; E')}^2 \| C^* \theta \|_{E'}^2 \\
\leq M^2 \| C^* \theta \|_{E'}^2,
\]

with \( M > 0 \), because the semi-group is bounded on \([0, T]\).

Since (5) is exactly controllable, there exists \( \gamma_1 > 0 \) such that
\[
\| C^* \theta \|_{E'} \leq \gamma_1 \| B^* S^*(T - .) C^* \theta \|_{L^2(0,T; U')}
\]

consequently, there exists \( \gamma = M(\gamma_1)^2 > 0 \) such that
\[
\| S^*(T - .) C^* \theta \|_{L^2(0,T; E')}^2 \leq \gamma \| B^* S^*(T - .) C^* \theta \|_{L^2(0,T; U')}^2
\]

and the result derives from proposition 3.2. \( \square \)

**Proposition 4.2**

If (5) is weakly controllable on \([0, T]\), then (3) + (4) is weakly remediable on \([0, T]\).

**Proof**

Using (20),(26) and (22) we have (3) + (4) weakly remediable if and only if \( \ker(H^* C^*) = \ker(C^*) \), this equivalent to \( \ker(H^* C^*) \subset \ker(C^*) \).

Let \( \theta \in \ker(H^* C^*) \), we have \( H^* C^* \theta = 0 \) then \( C^* \theta = 0 \), because \( \ker(H^*) = \{0\} \), hence \( \theta \in \ker(C^*) \) and the result follows from proposition 3.4. \( \square \)

In multi-actuators and multi-sensors cases, we have the following corollary.

**Corollary 4.3**

Strategic actuators are necessarily efficient.

The converse is not true (paragraphe 6).

### 5 Exact remediability with minimal energy

In this section, we consider the following exact remediability problem.

For a given \( f \in L^2(0, T; L^2(\Omega)) \), that is \( \tilde{f} \in L^2(0, T; \{0\} \times L^2(\Omega)) \), does a control \( u \in L^2(0, T; \mathcal{U}) \) such that
\[
y_{u,f}(T) = C H u + R \tilde{f} = 0
\]

and if \( u \) exists, is it optimal?

For \( \theta \in Y^* \equiv Y \), let
\[
\| \theta \|_{\mathcal{F}} = \left[ \int_0^T \| B^* S^*(T - s) C^* \theta \|_{U'}^2 \ ds \right]^\frac{1}{2}
\]

\( \| . \|_{\mathcal{F}} \) defines a semi-norm, and we have

**Lemma 5.1**

If \( \ker(C^*) = \{0\} \), there is equivalence between

(i) \( (S_T) + (E) \) is weakly remediable on \([0, T]\).

(ii) \( \ker(H^* C^*) = \{0\} \)

(iii) \( . \| . \|_{\mathcal{F}} \) is a norm on \( Y \).

**Proof**

(i) \( \iff \) (ii) derives from (20), proposition 3.4 and (26).

(ii) \( \implies \) (iii) Let \( \theta \in Y \) such that \( \| \theta \|_{\mathcal{F}} = 0 \)

The mapping \( s \mapsto \| B^* S^*(T - s) C^* \theta \|^2 \) is continuous and positive, we have \( \| B^* S^*(T - s) C^* \theta \|^2 = 0, \ \forall s \in [0, T] \)

then \( B^* S^*(T - .) C^* \theta = 0 \), i.e. \( H^* C^* \theta = 0 \). Since \( \ker(H^* C^*) = \{0\} \), we have \( \theta = 0 \).

(iii) \( \implies \) (ii) Derives from the fact that
\[
H^* C^* = B^* S^*(T - .) C^*
\]
Let the operator \( \Lambda = CHH^*C^* \). For \( \theta \in Y^* \equiv Y \), we have
\[
\Lambda \theta = \int_0^T CS(T-s)BB^*S^*(T-s)C^*\theta ds \in Y
\]
(32)

(iii) Let \( \theta \in \mathcal{F} \), we consider the linear mapping \( \Lambda \theta : \sigma \in Y \mapsto < \Lambda \theta, \sigma >_Y \in \mathbb{R} \), then we have
\[
|(< \Lambda \theta, \sigma >_Y)| = |< \Lambda \theta, \sigma >_Y| = |< \theta, \sigma >_Y| 
\leq \| \theta \|_Y \| \sigma \|_Y
\]
and then \( \Lambda \theta \) is continuous on \( Y \) for the topology of \( \mathcal{F} \), then it can be extended continuously to \( \mathcal{F} \), hence \( \Lambda \theta \in \mathcal{F}' \) and
\[
< \Lambda \theta, \sigma >_Y = < \theta, \sigma >_Y, \forall \sigma \in Y
\]
implies \( \| \Lambda \theta \|_{\mathcal{F}'} = \| \theta \|_Y \).

The operator \( \Lambda : \mathcal{F} \mapsto \mathcal{F}' \) is linear and surjective, using Riesz theorem.

\( \Lambda \) is also injective; Indeed, let \( \theta \in \mathcal{F} \) such that \( \Lambda \theta = 0 \), we have \( < \Lambda \theta, \theta > = 0 \), i.e. \( \| \theta \|_Y = 0 \) then \( \theta = 0 \). \( \Lambda \) is then an isomorphism from \( \mathcal{F} \) to \( \mathcal{F}' \).

Concerning the problem of exact remediability with minimal energy, we have the following result.

**Proposition 5.5**

If the observation \( y_f = R\tilde{f} \in \mathcal{F}' \), then there exists a unique element \( \tilde{f} \in \mathcal{F} \) such that \( \Lambda \tilde{f} = -y_f \), and the control
\[
u_{\tilde{f}}(t) = B^*S^*(T-t)C^*\tilde{f}
\]
(38)

verifies
\[
CH\nu_{\tilde{f}} + y_f = 0
\]
(39)

and \( \nu_{\tilde{f}} \) is optimal, with
\[
\| \nu_{\tilde{f}} \|_{L^2(\sigma(0,T);U)} = \| \tilde{f} \|_{\mathcal{F}}
\]
(40)

**Proof**

We have
\[
\Lambda \theta_{\mathcal{F}} = \int_0^T CS(T-s)BB^*S^*(T-s)C^*\theta ds
= \int_0^T CS(T-s)Bu_{\theta_i} ds = CHu_{\theta_i} = -y_f
\]

On other hand, we consider the set
\[
\mathcal{V} = \{ u \in L^2(0,T; U) : CHu + y_f = 0 \}
\]

\( \mathcal{V} \) is nonempty \( (u_{\theta_i} \in \mathcal{V}) \), closed and convex. We consider the function
\[
J(u) = \| CHu + y_f \|^2 + \| u \|^2
\]

For \( u \in \mathcal{V} \), we have \( J(u) = \| u \|^2 \), \( J \) is strictly convex on \( \mathcal{V} \), then it admits a unique minimum \( u^* \in \mathcal{V} \), which characterized by
\[
< u^*, v-u^* > \geq 0 ; \ \forall v \in \mathcal{V}.
\]

For \( v \in \mathcal{V} \), we have
\[
< u_{\theta_i}, v-u_{\theta_i} >_{L^2(0,T;U)} = \int_0^T < B^*S^*(T-t)\tilde{f}, v(t)-u_{\theta_i}(t) > dt
= < \tilde{f}, \int_0^T v(t) dt - \int_0^T CS(T-s)Bu_{\theta_i}(t) dt >
= < \tilde{f}, CHv - CHu_{\theta_i} > = < \tilde{f}, -y_f + y_f > = 0.
\]
Since \( u^* \) is unique, we have \( u^* = u_{\theta_f} \), and
\[
\| u_{\theta_f} \|_{L^2(0,T;U)}^2 = \int_0^T \| u_{\theta_f}(t) \|_U^2 \, dt = \int_0^T \| B^*S^*(T-t)C^*\theta_f \|_2^2 \, dt = \| \theta_f \|_{\mathcal{F}}^2.
\]
\( \square \)

We consider the set
\[
\mathcal{W} = \{ \bar{f} \in L^2(0,T;\{0\} \times X) / \exists u \in L^2(0,T;U) \text{ verifies } CHu + R\bar{f} = 0 \} \tag{41}
\]
The set \( \mathcal{W} \) is characterized by

**Proposition 5.6**

\( \mathcal{W} \) is the reciprocal image by \( R \) of \( \mathcal{F}' \), i.e.
\[
R\mathcal{W} = \mathcal{F}' \tag{42}
\]

**Proof**

For \( y \in \mathcal{F}' \), there exists a unique \( \theta \in \mathcal{F} \) such that \( \Lambda^*\theta = y \), let
\[
\int_0^T CS(T-s)BB^*S^*(T-s)C^*\theta ds = y
\]
Let \( u \) is the control defined by
\[
u(\cdot) = B^*S^*(T-\cdot)C^*\theta \in L^2(0,T;U)
\]
we have \( \Lambda^*\theta = \int_0^T CS(T-s)Bu(s)ds = y \), i.e. \( CHu = y \),
and for \( \bar{f} = -Bu \in L^2(0,T;\{0\} \times X) \), we have \( CHu = -RF = y \), then \( y \in R\mathcal{W} \).

Conversely, let \( y \in R\mathcal{W} \), there exists \( \bar{f} \in L^2(0,T;\{0\} \times L^2(\Omega)) \) such that \( y = RF \bar{f} \) and \( CHu + R\bar{f} = 0 \) with \( u \in L^2(0,T;U) \).

If we identify \( CHu \) to the linear mapping, \( L : \theta \in Y \mapsto CHu, \theta > \), we have
\[
L(\theta) = < CHu, \theta > = < C \int_0^T S(T-s)Bu(s)ds, \theta > = \int_0^T < u(s), B^*S^*(T-s)C^*\theta > ds
eq 0
\]
and using (31), we have
\[
|L(\theta)| \leq \| u \|_{L^2(0,T;U)} \| \theta \|_{\mathcal{F}}
\]

\( L \) is then a continuous linear mapping on \( Y \) for the topology of \( \mathcal{F} \), and hence has a unique continuous extension to \( \mathcal{F} \).
Hence \( L \in \mathcal{F}' \) and \( CHu = -RF \bar{f} = -y \in \mathcal{F}' \), therefore \( y \in \mathcal{F}' \). \( \square \)

### 6 Application

Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^n \), with a sufficiently regular boundary \( \partial \Omega \). We consider the following hyperbolic system
\[
\begin{align*}
\frac{\partial^2 x}{\partial t^2} &= \Delta x + \sum_{i=1}^{p} g_i(\xi)u_i(t) & \quad \text{in } \Omega \times [0,T] \\
x(\xi, 0) &= \frac{\partial x}{\partial \eta}(\xi, 0) = 0 & \quad \text{in } \Omega \\
x(\eta, t) &= 0 & \quad \text{on } \partial \Omega \times [0,T]
\end{align*}
\] (43)

If (43) is disturbed by \( f \in L^2(0,T;L^2(\Omega)) \), it is equivalent to
\[
\begin{align*}
\frac{\partial z}{\partial t}(\xi, t) &= Az(\xi, t) + (0, \sum_{i=1}^{p} g_i(\xi)u_i(t))^{tr} & \quad \text{in } \Omega \times [0,T] \\
(0, f(\xi, t))^{tr} &= 0 & \quad \text{in } \Omega \\
z(\xi, 0) &= 0 & \quad \text{in } \Omega
\end{align*}
\] (44)

(44) is augmented by the output equation
\[
y = Cz \tag{45}
\]

#### 6.1 One dimensional case: \( \Omega = [0, \alpha] \)

The eigenvectors of \( \Delta \) are defined by
\[
\varphi_n(\xi) = \sqrt{\frac{2}{\alpha}} \sin\left(\frac{n\pi \xi}{\alpha}\right) ; \quad \forall \ n \geq 1
\]
The eigenvalues associated are simple, and given by
\[
\lambda_n = -n^2 \left(\frac{\pi}{\alpha}\right)^2 \quad ; \quad \forall \ n \geq 1
\]
In the case of sensors \( (D,h) \) et \( (D',k) \), with \( D = supp(h) \subset ]0,1[ \) and \( D' = supp(k) \subset ]0,1[ \). Let \( n_0 \) such that \( h, \varphi_n > x \neq 0 \) and \( k, \varphi_n > x \neq 0 \).
Using proposition 3.8 and for \( T \geq 2\alpha \), an actuator \( (\Omega_1, g_1) \) is efficient if
\[
\int_{\Omega_1} g_1(\xi) \sin\left(\frac{n_0\pi \xi}{\alpha}\right) d\xi \neq 0 \tag{46}
\]
then for example, for \( g_1 = \varphi_{n_0} \), \( (\Omega_1, g_1) \) is efficient but not strategic, because the condition
\[
\int_{\Omega_1} g_1(\xi) \sin\left(\frac{n\pi \xi}{\alpha}\right) d\xi \neq 0 ; \quad \forall \ n \geq 1 \tag{47}
\]

is not verified.

**Remark 6.1**

If \( T \geq 2\alpha \) [6,7], the functions \( \sin\left(\frac{n\pi}{\alpha}\right)_{n \geq 1} \) and \( \cos\left(\frac{n\pi}{\alpha}\right)_{n \geq 0} \) constitutes a complete orthogonal system in \( L^2(0, \alpha) \). This is used to characterize efficient (respectively strategic) actuators.
6.2 Rectangle case: $\Omega = [0, \alpha [x]0, \beta [y]$

The eigenvectors of $\Delta$ are defined by

$$\varphi_{m,n}(\xi_1, \xi_2) = \frac{2}{\sqrt{\alpha \beta}} \sin\left(\frac{m\pi \xi_1}{\alpha}\right) \sin\left(\frac{n\pi \xi_2}{\beta}\right) ; \forall \ m, n \geq 1$$

The associated eigenvalues are

$$\lambda_{m,n} = -(\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2})\pi^2 ; \forall \ m, n \geq 1$$

We know that [7]

i) If $\frac{\alpha^2}{\beta^2} \notin Q$, then the eigenvectors are sample, then a single actuator $(\Omega_1, g_1)$ with $\Omega_1 = supp(g_1) \subset \Omega$, may be sufficient to have weak controllability.

Indeed, $(\Omega_1, g_1)$ is strategic if and only if

$$< g_1, \varphi_{m,n} >_{\Omega} \neq 0 ; \forall \ m, n \geq 1$$

i.e. $\forall \ m, n \geq 1$

$$\int_{\Omega_1} g_1(\xi_1, \xi_2) \sin\left(\frac{m\pi \xi_1}{\alpha}\right) \sin\left(\frac{n\pi \xi_2}{\beta}\right) d\xi_1 d\xi_2 \neq 0 ; \quad (48)$$

ii) If $\alpha = \beta = 1$, i.e. in a case of a square domain, we have $\lambda_{m,n} = -(m^2 + n^2)\pi^2$ and $\sup_{m,n \geq 1} r_{m,n} = \infty$, and then we cannot have the weak controllability by a finite number of actuators.

On the other hand, using corollary 3.9, (3) + (4) is weakly remediable if there exists $m_0, n_0 \geq 1$ such that

$$rg(M_{n_0}G^r_{m_0,1}) = q_1 \quad \text{and} \quad rg(M_{n_0}G^r_{m_0,0}) = q_2$$

Henceforth, in the case of sensors $(D, h)$ et $(D', k)$, an actuator is sufficient, for any $\alpha$ and $\beta$.

Indeed, $(\Omega_1, g_1)$ is efficient, if there exists $m_0, n_0 \geq 1$ such that

$$\sum_{j=1}^{r_{m_0,n_0}} < g_1, \varphi_{m_0,n_0,j} >_{\Omega} h, \varphi_{m_0,n_0,j} >_{\Omega} \neq 0$$

$$\sum_{j=1}^{r_{m_0,n_0}} < g_1, \varphi_{m_0,n_0,j} >_{\Omega} k, \varphi_{m_0,n_0,j} >_{\Omega} \neq 0 \quad (49)$$

that is the case if for example $g_1 = \varphi_{m_0,j_0,n_0,j_0}$ where $m_0,j_0,n_0,j_0$ are such that $< h, \varphi_{m_0,j_0,n_0,j_0} >_{\Omega} \neq 0$ and $< k, \varphi_{m_0,j_0,n_0,j_0} >_{\Omega} \neq 0$.

The actuator $(\Omega_1, g_1)$ is then efficient, but not strategic.

Remark 6.2

i) In the case where $\frac{\alpha^2}{\beta^2} \notin Q$, we have $r_{m,n} = 1, \forall \ m, n \geq 1$, the condition (49) becomes

$$< g_1, \varphi_{m_0,n_0} >_{\Omega} < h, \varphi_{m_0,n_0} >_{\Omega} \neq 0 \quad (50)$$

$$< g_1, \varphi_{m_0,n_0} >_{\Omega} > k, \varphi_{m_0,n_0} >_{\Omega} \neq 0$$

One actuator may be efficient, but a finite number of actuators cannot be strategic in finite time.

Thus, for example for $h$ such that

$$< h, \varphi_{m_0,j_0,n_0,j_0} >_{\Omega} \neq 0$$

and $k$ such that

$$< k, \varphi_{m_0,j_0,n_0,j_0} >_{\Omega} \neq 0$$

for $g_1 = \varphi_{m_0,j_0,n_0,j_0}$, the actuator $(\Omega_1, g_1)$ is efficient.

References


