On linear systems disturbed by wide-band noise

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Abstract

An approach, based on a distributed delay, is presented to handle wide-band noise processes. This approach is closely related with the semigroups of right translation and allows a reduction of a wide-band noise driven linear system to a white noise driven linear system. This reduction is applied to the Kalman-Bucy filtering in order to modify it to a wide-band noise driven linear system.

1 Introduction

The modern stochastic optimal control and filtering theories use white noise driven systems. The results such as the separation principle and the Kalman-Bucy filtering are based on the white noise model. Indeed, white noise, being a mathematical idealization, gives only an approximate description to actual noise. In some fields the parameters of actual noise are near to the parameters of white noise and so the mathematical methods of control and filtering for white noise driven systems can be satisfactorily applied in them. But in many fields white noise is a grude approximation to actual noise. Consequently, the theoretical optimal controls and the theoretical optimal estimators for white noise driven systems become not optimal and, indeed, might be quite far from being optimal. It becomes important to develop the problems of systems theory for a noise model that describe actual noise more adequately.

The issue is that actual noise is marked with a property which ensures the correlation of its values within a small time interval, i.e. if we denote such noise by \( \varphi \), then

\[
\text{cov} [\varphi(t + s), \varphi(t)] = \begin{cases} \Lambda(t, s) & \text{if } 0 \leq s < \varepsilon \\ 0 & \text{otherwise,} \end{cases}
\]

(1)

where \( t \geq 0 \) and \( s \geq 0 \) are time moments, \( \varepsilon > 0 \) is a small value and \( \Lambda \) is a nonzero function. A random process \( \varphi \) with the property (1) is called a wide-band noise process. The function \( \Lambda \) is called an autocovariance function. The wide-band noise process \( \varphi \) is said to be stationary if the autocovariance function \( \Lambda \) depends only on \( s \) (see [1]). If \( \varepsilon \) is so small that it is normally assumed to be 0, then the wide-band noise process \( \varphi \) is transformed into a white noise process. As it was mentioned, in many fields such substitution gives rise to tangible distortions.

There are different techniques of handling and working with the wide-band noise processes. For example, in [2] the approximate approach is used. Another approach, based on a representation of wide-band noise as a distributed delay of white noise, is suggested in [3].

In [3, 4] the Kalman-Bucy estimation results are modified to a wide-band noise driven systems. The proofs of respective results are based on the duality principle and, technically, they are routine being useless if one tries to modify them to nonlinear systems. It becomes important to develop a more handle technique of working with the wide-band noise processes represented through a distributed delay.

Some attempts in this way are made in [5] where a reduction of a wide-band noise driven system to a white noise driven system is presented. The reduction from [5] requires an unbounded observation system which does not allow to obtain a complete set of formulae for the respective optimal filter.

In this paper we establish another reduction of a wide-band noise driven system to a white noise driven system which does not require an unbounded observation system. We apply this reduction to obtain a complete set of formulae for the respective optimal filter. This reduction gives hope to push on studying nonlinear systems disturbed by wide-band noise.

2 Wide-band noise

Consider the random process

\[
\varphi(t) = \int_{\max(0,t-\varepsilon)}^{t} \phi(s-t) \, dw(s), \quad t \geq 0,
\]

(2)

where \( \varepsilon > 0 \), \( \phi \in L_2(-\varepsilon,0) \) and \( w \) is a stationary process with orthogonal increments satisfying \( \text{cov} w(t) = t \) and \( Ew(t) = 0 \) for all \( t \geq 0 \). One can easily verify that the random process \( \varphi \), as defined by (2), is a wide-band noise
process which becomes stationary starting the time moment \( t = \varepsilon \) with the autocovariance function

\[
\lambda(s) = \int_{-\varepsilon}^{s} \phi(r) \phi(r + s) \, dr, \quad 0 \leq s \leq \varepsilon.
\] (3)

Formally, the wide-band noise process (2) can be interpreted as a vibration. At moment \( t \) a vibration that is formed by the action of white noise during the time between \( t - \varepsilon \) and \( t \) affects the system. Values of white noise until \( t - \varepsilon \) do not take part in the formation of this vibration at moment \( t \), because their weight is sufficiently small and we can neglect them in the model (2). Consequently, the function \( \phi \) stands for the coefficient of relaxing the initial effect at different time moments. By this interpretation, the model (2) corresponds to real cases when the vibration generated by white noise stands to affect a system starting with the initial state \( \xi = 0 \) and \( \eta = 0 \), because their weight is sufficiently small and we can neglect them in the model (2). Thus, given a function \( \lambda \) in \( L_2(0, \varepsilon) \), there is a variety of wide-band noise processes in the form (2) which have the same autocovariance function \( \lambda \) starting the time moment \( \varepsilon \). This variety is reached by different solutions of the equation (3) and also by different processes \( w \).

## 3 Notation

Let \( X, Z, H, H_1, H_2 \) and \( G \) be separable Hilbert spaces, let \( L(X, Z) \) be the space of linear bounded operators from \( X \) to \( Z \) and let \( B_2(a, b; L(X, Z)) \) be the space of equivalence classes of strongly Lebesgue measurable and square integrable operator-valued functions from \( [a, b] \) to \( L(X, Z) \). From the discussion given in Section 2, it is reasonable to define a Hilbert space version of the representation (2) and, hence, we let \( \varphi_1 \) and \( \varphi_2 \) be \( X \)- and \( Z \)-valued wide-band noise processes in the form

\[
\varphi_1(t) = \int_{\max(0, t - \varepsilon)}^{t} \Phi_1(s - t) \, dw_1(s), \quad t \geq 0,
\] (5)

\[
\varphi_2(t) = \int_{\max(0, t - \delta)}^{t} \Phi_2(s - t) \, dw_2(s), \quad t \geq 0,
\] (6)

where \( \varepsilon > 0 \), \( \delta > 0 \), \( w_1 \) and \( w_2 \) are \( H_1 \)- and \( H_2 \)-valued Wiener processes, \( \Phi_1 \) and \( \Phi_2 \) are strongly differentiable operator-valued functions from \([-\varepsilon, 0]\) to \( L(H_1, X) \) and \( L(H_2, Z) \), respectively, with

\[
\frac{d}{d\theta} \Phi_1 \in B_2(-\varepsilon, 0; L(H_1, X)), \quad \Phi_1(-\varepsilon) = 0,
\]

\[
\frac{d}{dx} \Phi_2 \in B_2(-\delta, 0; L(H_2, Z)), \quad \Phi_2(-\delta) = 0.
\]

Note that by the interpretation of the wide-band noise process (2) given above, the function \( \phi \) is the coefficient of relaxing the initial effect at distinct time moments. Therefore, it should be a decreasing (hence, almost everywhere differentiable) function with \( \phi(-\varepsilon) = 0 \). In view of this, the conditions on \( \Phi_1 \) and \( \Phi_2 \) are reasonable.
Consider the partially observable linear system
\[ dx(t) = (Ax(t) + \varphi_1(t))dt + \Phi dw(t), \]
\[ x(0) = x_0, \quad t \geq 0, \]  
\[ dz(t) = (Cx(t) + \varphi_2(t))dt + \Psi dv(t), \]
\[ z(0) = 0, \quad t \geq 0, \]  
(7)

where \( x \) is a state process, \( z \) is an observation process, \( A \) is the infinitesimal generator of a strongly continuous semi-group \( \mathcal{U} \) of linear bounded operators on \( X \), and \( \Phi \in \mathcal{L}(H, X) \), \( \Psi \in \mathcal{L}(G, Z) \), and \( \varphi_1 \) and \( \varphi_2 \) are wide-band noise processes defined by (5) and (6), \( x_0 \) is an \( X \)-valued Gaussian random variable, \( w \) and \( v \) are \( H \)- and \( G \)-valued Wiener processes so that \( x_0, (w, w_1, w_2) \) and \( v \) are mutually independent, and \( w, w_1 \) and \( w_2 \) are correlated. In integral form, the system (7)–(8) can be represented as it is shown below:
\[ x(t) = \mathcal{U}(t)x_0 + \int_0^t \mathcal{U}(t-s)\varphi_1(s)ds \]
\[ + \int_0^t \mathcal{U}(t-s)\Phi dw(s), \quad t \geq 0, \]
\[ z(t) = \int_0^t (Cz(s) + \varphi_2(s))ds + \Psi v(t), \quad t \geq 0. \]

In this paper we will reduce the system (7)–(8), disturbed by wide-band and white noise processes, to the system which is disturbed by purely white noise. For this, we set some special notations which will be used throughout this paper.

Let \( \hat{X} = L_2(-\varepsilon, 0; X) \) (the space of strongly measurable and square integrable vector-valued functions from \([\varepsilon, 0] \) to \( X \)) and let \( \hat{Z} = L_2(-\delta, 0; Z) \). We will consider the semigroups of right translation \( T_1 \in \mathcal{S}(\hat{X}) \) and \( T_2 \in \mathcal{S}(\hat{Z}) \) defined by
\[ [T_1(t)f](\theta) = \begin{cases} f(\theta - t), & \theta - t \geq -\varepsilon \\ f, & 0 \leq \theta < -\varepsilon \end{cases}, \]
a.e. \( \theta \in [-\varepsilon, 0] \), \( t \geq 0, \) \( f \in \hat{X} \),
\[ [T_2(t)g](\theta) = \begin{cases} g(\alpha - t), & \alpha - t \geq -\delta \\ g, & -\delta < \alpha < 0 \end{cases}, \]
a.e. \( \alpha \in [-\delta, 0] \), \( t \geq 0, \) \( g \in \hat{Z} \),

and the linear integral operators \( \Gamma_1 \) and \( \Gamma_2 \) from \( \hat{X} \) and \( \hat{Z} \) to \( X \) and \( Z \), respectively, defined by
\[ \Gamma_1 f = \int_{-\varepsilon}^0 f(\theta)d\theta, \quad f \in \hat{X}, \]
\[ \Gamma_2 g = \int_{-\delta}^0 g(\alpha)d\alpha, \quad g \in \hat{Z}. \]

The infinitesimal generators of \( T_1 \) and \( T_2 \) are the differential operators \( -d/d\theta \) and \( -d/d\alpha \) operating from
\[ D(-d/d\theta) = \{ f \in \hat{X} : (-d/d\theta)f \in \hat{X}, \quad f(-\varepsilon) = 0 \} \]
and
\[ D(-d/d\alpha) = \{ g \in \hat{Z} : (-d/d\alpha)g \in \hat{Z}, \quad g(-\delta) = 0 \}. \]

to \( \hat{X} \) and \( \hat{Z} \), respectively. We will use the notation
\[ \hat{A} = \begin{bmatrix} A & \Gamma_1 & 0 \\ 0 & -d/d\theta & 0 \\ 0 & 0 & -d/d\alpha \end{bmatrix}. \]

One can easily verify that
\[ \hat{A}^{*} = \begin{bmatrix} A^* & 0 & 0 \\ \Gamma_1^* & d/d\theta & 0 \\ 0 & 0 & d/d\alpha \end{bmatrix}, \]
where \( A^* \) is the adjoint of \( A \), \( d/d\theta \) and \( d/d\alpha \) are differential operators from
\[ D(d/d\theta) = \{ f \in \hat{X} : (d/d\theta)f \in \hat{X}, \quad f(0) = 0 \} \]
and
\[ D(d/d\alpha) = \{ g \in \hat{Z} : (d/d\alpha)g \in \hat{Z}, \quad g(0) = 0 \} \]
to \( \hat{X} \) and \( \hat{Z} \), respectively. From [6], it follows that \( \hat{A} \) is the infinitesimal generator of the strongly continuous semigroup
\[ \mathcal{U}(t) = \begin{bmatrix} \mathcal{U}(t) & \mathcal{E}(t) & 0 \\ 0 & T_1(t) & 0 \\ 0 & 0 & T_2(t) \end{bmatrix}, \quad t \geq 0, \]
where \( \mathcal{E} : [0, \infty) \to \mathcal{L}(\hat{X}, X) \) is the function defined by
\[ \mathcal{E}(t)h = \int_{\max(-\varepsilon, -t)}^{0} \mathcal{U}(t+r)h(r)dr, \quad t \geq 0, \quad h \in \hat{X}. \]

We will denote by \( \hat{\Phi} \) the operator
\[ \hat{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ 0 & \hat{\Phi}_1 & 0 \\ 0 & 0 & \hat{\Phi}_2 \end{bmatrix} \in \mathcal{L}(H \times H_1 \times H_2, X \times \hat{X} \times \hat{Z}), \]

where the operators \( \hat{\Phi}_1 \) and \( \hat{\Phi}_2 \) are defined by
\[ \hat{\Phi}_1 h(\theta) = \frac{d}{d\theta}\Phi_1(\theta)h, \quad \text{a.e. } \theta \in [-\varepsilon, 0], \quad h \in H_1, \]
\[ \hat{\Phi}_2 h(\alpha) = \frac{d}{d\alpha}\Phi_2(\alpha)h, \quad \text{a.e. } \alpha \in [-\delta, 0], \quad h \in H_2, \]
and by \( \hat{w} \) the \( H \times H_1 \times H_2 \)-valued Wiener process
\[ \hat{w}(t) = \begin{bmatrix} w(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}, \quad t \geq 0. \]

Clearly, \( \hat{\Phi}_1 \in \mathcal{L}(H_1, \hat{X}) \) and \( \hat{\Phi}_2 \in \mathcal{L}(H_2, \hat{Z}). \)

Also, we will use the operators
\[ \hat{C} = \begin{bmatrix} C & 0 & \Gamma_2 \end{bmatrix} \in \mathcal{L}(X \times \hat{X} \times \hat{Z}, \hat{Z}) \]
and
\[ \hat{I} = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \in \mathcal{L}(X \times \hat{X} \times \hat{Z}, X), \]
where \( I \) is the identity operator on \( X \).
4 Reduction

Let \( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \) be \( \hat{X} \)- and \( \hat{Z} \)-valued random processes defined by

\[
\begin{align*}
\hat{\varphi}_1(t)(\theta) &= \int_{\max(0,t-\varepsilon)}^{t} \frac{d}{d\theta} \Phi_1(s - t + \theta) \, dw_1(s), \\
\text{a.e. } \theta &\in [-\varepsilon,0], \ t \geq 0, \\
\hat{\varphi}_2(t)(\alpha) &= \int_{\max(0,t-\delta-\alpha)}^{t} \frac{d}{d\alpha} \Phi_2(s + \alpha) \, dw_2(s), \\
\text{a.e. } \alpha &\in [-\delta,0], \ t \geq 0.
\end{align*}
\]  
(9)

Lemma 1. The random processes \( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \), defined by (9) and (10), have the representations

\[
\hat{\varphi}_1(t) = \int_{0}^{t} T_1(t - s) \hat{\Phi}_1 dw_1(s), \ t \geq 0,
\]
(11)

\[
\hat{\varphi}_2(t) = \int_{0}^{t} T_2(t - s) \hat{\Phi}_2 dw_2(s), \ t \geq 0.
\]
(12)

Proof. Let us prove the representation in (11) for \( \hat{\varphi}_1 \). For \( h \in H_1 \), we have

\[
\begin{align*}
\left[ T_1(t - s) \hat{\Phi}_1 h \right](\theta) &= \left\{ \begin{array}{l}
\hat{\Phi}_1 h(s - t + \theta), \ s - t + \theta \geq -\varepsilon \\
0, \ s - t + \theta < -\varepsilon
\end{array} \right. \\
&= \left\{ \begin{array}{l}
\frac{d}{d\theta} \Phi_1(s - t + \theta) h(s), \ s - t + \theta \geq -\varepsilon \\
0, \ s - t + \theta < -\varepsilon
\end{array} \right.
\end{align*}
\]

Therefore,

\[
\hat{\varphi}_1(t)(\theta) = \int_{\max(0,t-\varepsilon)}^{t} \frac{d}{d\theta} \Phi_1(s - t + \theta) \, dw_1(s)
\]

\[
= \left[ \int_{0}^{t} T_1(t - s) \hat{\Phi}_1 dw_1(s) \right](\theta).
\]

The representation in (12) for \( \hat{\varphi}_2 \) can be proved in a similar way.

Lemma 2. For \( \varphi_1 \) and \( \varphi_2 \), defined by (5) and (6), and for \( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \), defined by (9) and (10),

\[
\varphi_1(t) = \Gamma \hat{\varphi}_1(t), \ \varphi_2(t) = \Gamma \hat{\varphi}_2(t), \ t \geq 0.
\]
(13)

Proof. The first equality in (13) follows from

\[
\begin{align*}
\Gamma \hat{\varphi}_1(t) &= \int_{-\varepsilon}^{0} \hat{\varphi}_1(t)(\theta) \, d\theta \\
&= \int_{-\varepsilon}^{0} \int_{\max(0,t-\varepsilon-\theta)}^{t} \frac{d}{d\theta} \Phi_1(s - t + \theta) \, dw_1(s) \, d\theta \\
&= \int_{\max(0,t-\varepsilon)}^{t} \frac{d}{d\theta} \Phi_1(s - t + \theta) \, dw_1(s) \\
&= \int_{\max(0,t-\varepsilon)}^{t} \Phi_1(s - t) \, dw_1(s) = \varphi_1(t).
\end{align*}
\]

The second equality in (13) can be proved in a similar way.

By Lemmas 1 and 2, the random processes \( x, \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \), defined by (7), (9) and (10), respectively, are mild solutions of the linear stochastic differential equations

\[
\begin{align*}
\frac{d}{dt} \hat{x}(t) &= (A \hat{x}(t) + \Gamma \hat{\varphi}_1(t)) \, dt + \Phi dw(t), \\
\frac{d}{dt} \hat{\varphi}_1(t) &= (d/d\theta) \phi(t) \, dt + \hat{\Phi} dw(t), \\
\frac{d}{dt} \hat{\varphi}_2(t) &= (d/d\theta) \phi(t) \, dt + \hat{\Phi} dw(t), \\
x(0) &= x_0, \ \hat{\varphi}_1(0) = 0, \ \hat{\varphi}_2(0) = 0, \ t \geq 0,
\end{align*}
\]

and the random process \( z \), defined by (8), satisfies

\[
\frac{d}{dt} z(t) = (C \hat{x}(t) + \Gamma_2 \hat{\varphi}_2(t)) \, dt + \Phi dw(t), \ z(0) = 0, \ t \geq 0.
\]

If

\[
\hat{x}(t) = \left[ \begin{array}{c}
\hat{\varphi}_1(t) \\
\hat{\varphi}_2(t)
\end{array} \right], \ t \geq 0,
\]

then using the notations from Section 3, for the random processes \( \hat{x} \) and \( z \), we obtain

\[
\begin{align*}
\frac{d}{dt} \hat{x}(t) &= \hat{A} \hat{x}(t) \, dt + \hat{\Phi} dw(t), \ \hat{x}(0) = \tilde{r} x_0, \ t \geq 0, \\
\frac{d}{dt} z(t) &= \tilde{C} \hat{x}(t) \, dt + \Phi dw(t), \ z(0) = 0, \ t \geq 0.
\end{align*}
\]

Thus, the system (7)–(8) disturbed by white and wide-band noise processes is reduced to the system (14)–(15) which is disturbed only by white noise. This reduction is significant as the operator \( \tilde{C} \) is bounded whereas in the reduction from [5] the respective operator is unbounded.

5 Application to filtering

In this section we apply the reduction from Section 4 to the filtering problem for the partially observable system (7)–(8). For, assume that the conditions and notations from Sections 3 and 4 hold and \( Z = R^k \) is the k-dimensional Euclidean space. Additionally, let \( \text{cov} x_0 = \Gamma_0 \) and \( \text{E} x_0 = 0 \). We denote \( \tilde{V} = \Psi \Psi^* \), where \( \Psi \) is the covariance operator of the Wiener process \( v \). Assume that \( V \) and \( \Psi \) are invertible. The covariance operator of \( \tilde{w} \) is

\[
\tilde{W} = \begin{bmatrix}
W & W_{01} & W_{02} \\
W_{01} & W_{11} & W_{12} \\
W_{02} & W_{12} & W_{22}
\end{bmatrix},
\]

where \( W \) and \( W_{ij} \) are the respective covariance operators of \( w, w_1 \) and \( w_2 \).

Then the best estimate \( \hat{x}(t) \) of \( \hat{x}(t) \) based on the observations \( z(s), 0 \leq s \leq t \), where \( (\hat{x}, z) \) is defined by (14)–(15), is a solution (in the mild sense) of the linear stochastic differential equation (see [7])

\[
\begin{align*}
\frac{d}{dt} \hat{x}(t) &= \hat{A} \hat{x}(t) \, dt + \hat{\Phi}(t) \tilde{C} \tilde{V}^{-1} (d \hat{x}(t) - \tilde{C} \hat{x}(t) \, dt), \\
\hat{x}(0) &= 0, \ t \geq 0,
\end{align*}
\]

where \( \hat{P} \) is a solution (in the scalar product sense) of the Riccati equation

\[
\begin{align*}
\frac{d}{dt} \hat{P}(t) &= \hat{A} \hat{P}(t) + \hat{A}^* \hat{P}(t) + \hat{\Phi} \hat{W} \hat{P}(t) \\
&\quad - \tilde{C} \hat{P}(t) \tilde{C}^* \tilde{V}^{-1} \tilde{C} \hat{P}(t), \ \hat{P}(0) = \tilde{r}^* \Gamma_0 \tilde{r}, \ t \geq 0.
\end{align*}
\]
Let \( \hat{x}(t) \) be the best estimate of \( x(t) \) based on the observations \( z(s), 0 \leq s \leq t \), where \((x, z)\) is defined by (7)–(8) and (5)–(6). Then \( \hat{x}(t) = T \hat{x}(t) \). Hence, an equation for \( \hat{x}(t) \) can be deduced from the equations (16) and (17). For that, we decompose \( \hat{P} \) in the form

\[
\hat{P}(t) = \begin{bmatrix}
\hat{R}_0(t) & \hat{R}_1(t) & \hat{R}_2(t) \\
\hat{P}_01(t) & \hat{P}_11(t) & \hat{P}_12(t) \\
\hat{P}_02(t) & \hat{P}_12(t) & \hat{P}_22(t)
\end{bmatrix} \in \mathcal{L}(X \times \hat{X} \times \hat{Z}).
\]

Since the values of \( \hat{P} \) are nuclear operators, the functions \( \hat{R}_01, \hat{R}_02, \hat{R}_{11}, \hat{R}_{12} \) and \( \hat{P}_{22} \) can be represented in the following integral form

\[
\hat{R}_01(t) = \int_{-\delta}^{\theta} \hat{R}_01(t, \theta) d\theta, \quad f \in \hat{X},
\]

\[
\hat{R}_02(t) = \int_{-\delta}^{\alpha} \hat{R}_02(t, \alpha) d\alpha, \quad g \in \hat{Z},
\]

\[
[\hat{R}_1(t)f](\theta) = \int_{-\delta}^{\theta} \hat{R}_1(t, \theta, \tau) f(\tau) d\tau, \quad f \in \hat{X},
\]

\[
[\hat{R}_2(t)g](\alpha) = \int_{-\delta}^{\alpha} \hat{R}_2(t, \theta, \alpha) g(\alpha) d\alpha, \quad g \in \hat{Z},
\]

where \( \hat{R}_01, \hat{R}_02, \hat{R}_{11}, \hat{R}_{12} \) and \( \hat{P}_{22} \) are the respective kernels being square integrable functions in \( \theta, \tau \in [-\epsilon, 0] \) and in \( \alpha, \sigma \in [-\delta, 0] \) under fixed \( t \geq 0 \). Let

\[
P_{01}(t, \theta) = \int_{-\delta}^{\theta} \hat{R}_01(t, \theta_1) d\theta_1,
\]

\[
P_{02}(t, \alpha) = \int_{-\delta}^{\alpha} \hat{R}_02(t, \alpha_1) d\alpha_1,
\]

\[
P_{11}(t, \theta, \tau) = \int_{-\delta}^{\theta} \int_{-\delta}^{\tau} \hat{R}_{11}(t, \theta_1, \tau_1) d\tau_1 d\theta_1,
\]

\[
P_{12}(t, \theta, \alpha) = \int_{-\delta}^{\theta} \int_{-\delta}^{\alpha} \hat{R}_{12}(t, \theta_1, \alpha_1) d\alpha_1 d\theta_1,
\]

\[
P_{22}(t, \alpha, \sigma) = \int_{-\delta}^{\alpha} \int_{-\delta}^{\sigma} \hat{P}_{22}(t, \alpha_1, \sigma_1) d\sigma_1 d\alpha_1,
\]

where \( \theta, \tau \in [-\epsilon, 0], \alpha, \sigma \in [-\delta, 0] \) and \( t \geq 0 \). We will derive an equation for \( \hat{x}(t) \) in terms of \( P_{00}, P_{01}, P_{02}, P_{11}, P_{12} \) and \( P_{22} \). Therefore, at first we present the equations for them.

**Theorem 2.** Let

\[
\begin{aligned}
M(t) &= P_{00}(t)C^* + P_{02}(t,0), \\
M_1(t, \theta) &= P_{01}(t, \theta)C^* + P_{12}(t, \theta, 0), \\
M_2(t, \alpha) &= P_{02}(t, \alpha)C^* + P_{22}(t, \alpha, 0),
\end{aligned}
\]

where \( \theta \in [-\epsilon, 0], \alpha \in [-\delta, 0] \) and \( t \geq 0 \). Then the functions \( P_{00}, P_{01}, \) and \( P_{02} \) are solutions (in the scalar product sense) of the equations

\[
\frac{d}{dt}P_{00}(t) = P_{00}(t)A^* + AP_{00}(t) + P_{01}(t, 0)
\]

\[
+ P_{01}(t, 0) + \Phi W \Phi^* - M(t)\hat{V}^{-1} M^*(t),
\]

\[
\begin{aligned}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \right) P_{01}(t, \theta) &= AR_{01}(t, \theta) + P_{11}(t, 0, \theta) \\
+ \Phi W_{01} \Phi^* - M(t)\hat{V}^{-1} M^*(t),
\end{aligned}
\]

\[
\begin{aligned}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) P_{02}(t, \alpha) &= AR_{02}(t, \alpha) + P_{22}(t, 0, \alpha) \\
+ \Phi W_{02} \Phi^* - M(t)\hat{V}^{-1} M^*(t),
\end{aligned}
\]

with the initial and boundary conditions

\[
\begin{aligned}
P_{00}(0) &= P_0, \\
P_{01}(0, \theta) &= 0, -\epsilon \leq \theta \leq 0, \\
P_{02}(0, \alpha) &= 0, -\delta \leq \alpha \leq 0, \\
P_{01}(t, -\epsilon) &= 0, P_{02}(t, -\delta) = 0, t \geq 0.
\end{aligned}
\]

**Sketch of proof.** The derivation of the equations (19)–(21) is routine and, therefore, here we give a brief sketch of that. Writing the Riccati equation (17) in componentwise form, we obtain the operator equations for \( P_{00}, \hat{P}_{01} \) and \( \hat{P}_{02} \) and, using the fact that \( \hat{P} \) is a scalar product solution of the Riccati equation (17), we deduce the equations for the kernels \( \hat{P}_{01} \) and \( \hat{P}_{02} \). Then, the equations (19)–(21) for \( P_{00}, P_{01} \) and \( P_{02} \) will be seen. The conditions in (22) follow from the initial condition in (17) and from the definition of the functions \( P_{01} \) and \( P_{02} \). The complete proof of this theorem can be found in [8].

**Theorem 3.** Under the notation (18), the functions \( P_{11}, P_{12}, \) and \( P_{22} \) are solutions (in the ordinary sense) of the equations

\[
\begin{aligned}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau} \right) P_{11}(t, \theta, \tau) &= M(t, \theta) \hat{V}^{-1} M^*(t, \tau), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \alpha} \right) P_{12}(t, \theta, \alpha) &= M(t, \theta) \hat{V}^{-1} M^*(t, \alpha), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \sigma} \right) P_{22}(t, \alpha, \sigma) &= M(t, \alpha) \hat{V}^{-1} M^*(t, \sigma),
\end{aligned}
\]

with the initial and boundary conditions

\[
\begin{aligned}
P_{11}(0, \theta, \tau) &= P_{11}(t, -\epsilon, \tau) = P_{11}(t, \theta, -\epsilon) = 0, \\
-\epsilon \leq \theta \leq 0, -\epsilon \leq \tau \leq 0, t \geq 0, \\
P_{12}(0, \theta, \alpha) &= P_{12}(t, -\epsilon, \alpha) = P_{12}(t, \theta, -\delta) = 0, \\
-\epsilon \leq \theta \leq 0, -\delta \leq \alpha \leq 0, t \geq 0, \\
P_{22}(0, \alpha, \sigma) &= P_{22}(t, -\delta, \sigma) = P_{22}(t, \alpha, -\delta) = 0, \\
-\delta \leq \alpha \leq 0, -\delta \leq \sigma \leq 0, t \geq 0.
\end{aligned}
\]

**Sketch of proof.** The derivation of the equations (23)–(25) is similar to the derivation of the equations (19)–(21). Writing the Riccati equation (17) in componentwise form, we obtain the operator equations for \( \hat{P}_{11}, \hat{P}_{12}, \) and \( \hat{P}_{22} \). Then, the
equations (23)-(25) for \( P_{11} \), \( P_{12} \) and \( P_{22} \) will be seen. The conditions in (26) are easy to verify. For the complete proof of Theorem 2, we again refer to [8].

**Theorem 4.** The best estimate \( \hat{x}(t) \) of \( x(t) \) based on the observations \( z(s), 0 \leq s \leq t \), exists and it is a solution (in the mild sense) of the equation

\[
d\hat{x}(t) = (A\hat{x}(t) + \hat{\varphi}_1(t,0))dt + (P_{00}(t,\theta)C^* + P_{12}(t,\theta,0)) \hat{V}^{-1}d\hat{z}(t),
\]

\[
\hat{x}(0) = 0, t \geq 0,
\]  

(27)

where the random process \( \hat{\varphi}_1 \) (called an innovation process) satisfies

\[
d\hat{\varphi}_1(t) = d\hat{z}(t) - C\hat{x}(t)dt - \hat{\varphi}_2(t,0)dt,
\]

\[
\hat{\varphi}_1(0) = 0, t \geq 0,
\]  

(28)

the random processes \( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \) are solutions (in the ordinary sense) of the equations

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \right) \hat{\varphi}_1(t, \theta)dt = (P_{01}(t, \theta)C^* + P_{12}(t, \theta, 0)) \hat{V}^{-1}d\hat{z}(t), \tag{29}
\]

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha} \right) \hat{\varphi}_2(t, \alpha)dt = (P_{02}(t, \alpha)C^* + P_{22}(t, \alpha, 0)) \hat{V}^{-1}d\hat{z}(t), \tag{30}
\]

satisfying the initial and boundary conditions

\[
\begin{aligned}
\hat{\varphi}_1(0, 0) &= 0, -\varepsilon \leq \theta \leq 0, \\
\hat{\varphi}_2(0, \alpha) &= 0, -\delta \leq \alpha \leq 0, \\
\hat{\varphi}_1(t, -\varepsilon) &= 0, \hat{\varphi}_2(t, -\delta) = 0, t \geq 0,
\end{aligned}
\]  

(31)

and \( P_{00}, P_{01}, P_{02}, P_{11}, P_{12}, P_{22} \) are solutions of the equations (19)–(21) and (23)–(25) satisfying the initial and boundary conditions in (22) and in (26), respectively.

**Proof.** Let

\[
\hat{\hat{x}}(t) = \begin{bmatrix} \hat{x}(t) \\ \hat{\varphi}_1(t) \\ \hat{\varphi}_2(t) \end{bmatrix}, t \geq 0.
\]

Denote

\[
\hat{\varphi}_1(t, \theta) = \int_{-\varepsilon}^{\theta} \left[ \hat{\varphi}_1(t, \tau) \right] d\tau, -\varepsilon \leq \theta \leq 0, t \geq 0,
\]

\[
\hat{\varphi}_2(t, \alpha) = \int_{-\delta}^{\alpha} \left[ \hat{\varphi}_2(t, \sigma) \right] d\sigma, -\delta \leq \alpha \leq 0, t \geq 0.
\]

Then for this functions, the conditions in (31) hold. Furthermore, we have

\[
\Gamma_1 \hat{\varphi}_1(t) = \int_{-\varepsilon}^{\theta} \frac{\partial}{\partial \theta} \hat{\varphi}_1(t, \theta) d\theta = \hat{\varphi}_1(t, 0) - \hat{\varphi}_1(t, -\varepsilon) = \hat{\varphi}_1(t, 0).
\]

In a similar way, \( \Gamma_2 \hat{\varphi}_2(t) = \hat{\varphi}_2(t, 0) \). Therefore,

\[
d\hat{\varphi}_1(t) = d\hat{z}(t) - C\hat{x}(t)dt - \hat{\varphi}_2(t, 0)dt = d\hat{z}(t) - C\hat{x}(t)dt - \Gamma_2 \hat{\varphi}_2(t)dt.
\]

Using this equality, we can write the equation (16) in component-wise form as follows:

\[
d\hat{x}(t) = (A\hat{x}(t) + \hat{\varphi}_1(t,0))dt + (P_{00}(t,\theta)C^* + \hat{\hat{\varphi}}_2(t,\theta)) \hat{V}^{-1}d\hat{z}(t),
\]

\[
\hat{x}(0) = 0, t \geq 0,
\]  

(32)

\[
d\hat{\varphi}_1(t) = -d_t \hat{\varphi}_1(t)dt + (P_{01}(t,\theta)C^* + \hat{\hat{\varphi}}_2(t,\theta)) \hat{V}^{-1}d\hat{z}(t),
\]

\[
\hat{\varphi}_1(0) = 0, t \geq 0,
\]  

(33)

\[
d\hat{\varphi}_2(t) = -d_{t\alpha} \hat{\varphi}_2(t)dt + (P_{02}(t,\alpha)C^* + \hat{\hat{\varphi}}_2(t,\alpha)) \hat{V}^{-1}d\hat{z}(t),
\]

\[
\hat{\varphi}_2(0) = 0, t \geq 0,
\]  

(34)

From

\[
\Gamma_2 \hat{\hat{\varphi}}_2(t)h = \int_{-\delta}^{\theta} \hat{\hat{\varphi}}_2(t,\alpha)h \, d\alpha
\]

\[
= \int_{-\delta}^{\theta} \frac{\partial}{\partial \alpha} \hat{\hat{\varphi}}_2(t,\alpha)h \, d\alpha
\]

\[
= P_{02}(t,\theta,0)h - P_{02}(t,-\delta)h = P_{02}(t,0)h, h \in X,
\]

it follows that \( \hat{\hat{\varphi}}_2(t) = P_{02}(t,0) \). Using this equality in (32), we obtain that \( \hat{x} \) is a mild solution of the equation (27). Furthermore, from

\[
\Gamma_2 \hat{\hat{\varphi}}_2(t) \hat{f} = \int_{-\delta}^{\theta} \int_{-\varepsilon}^{\tau} P_{02}(t,\theta,\alpha)h \, d\theta \, d\alpha
\]

\[
= \int_{-\delta}^{\theta} \int_{-\varepsilon}^{\tau} \left( \frac{\partial}{\partial \theta} P_{02}(t,\theta,\alpha) \right) f(\theta) \, d\theta \, d\alpha
\]

\[
= \int_{-\delta}^{\theta} \left( \frac{\partial}{\partial \theta} P_{02}(t,\theta,0) \right) f(\theta) \, d\theta, \ f \in \hat{X},
\]

it follows that

\[
\hat{\hat{\varphi}}_2(t) \Gamma_2^* h(\theta) = \left( \frac{\partial}{\partial \theta} P_{02}(t,\theta,0) \right)^* h
\]

\[
= \frac{\partial}{\partial \theta} P_{02}(t,\theta,0)h, h \in Z.
\]

Also, we have

\[
\hat{\hat{\varphi}}_2(t) \Gamma_2^* h(\theta) = \frac{\partial}{\partial \theta} P_{02}(t,\theta,0)h
\]

\[
= \frac{\partial}{\partial \theta} P_{02}(t,\theta,0)h, h \in X.
\]

Therefore,

\[
\left[ \hat{\hat{\varphi}}_2(t) \Gamma_2^* h(\theta) \right] = \frac{\partial}{\partial \theta} \hat{\hat{\varphi}}_2(t,\theta,0)h
\]

\[
= \frac{\partial}{\partial \theta} P_{02}(t,\theta,0)h, h \in Z.
\]
This implies
\[ \left[ T_1(t - s)(\tilde{P}_{01}(s)C^* + \tilde{P}_{12}(s)\Gamma_2^*)h\right](\theta) = \begin{cases} \left( \frac{\partial}{\partial \theta} \right) M_1(s, s - t +\theta)h, & s + \theta \geq t - \varepsilon \\ 0, & s + \theta < t - \varepsilon \end{cases}, \quad h \in \mathbb{Z}. \]

Using this equality in (33), we obtain
\[ \begin{array}{l}
\left[ \tilde{\tilde{\varphi}}_1(t) \right](\theta) \\
= \int_0^t T_1(t - s)(\tilde{P}_{01}(s)C^* + \tilde{P}_{12}(s)\Gamma_2^*)\tilde{V}^{-1}d\varepsilon(s) \left( \theta \right) \\
= \int_{\max(0, t - \varepsilon - \theta)}^t \frac{\partial}{\partial \theta} M_1(s, s - t +\theta)\tilde{V}^{-1}d\varepsilon(s). 
\end{array} \]

Hence,
\[ \tilde{\varphi}_1(t, \theta) = \int_{\varepsilon - \theta}^0 \left[ \tilde{\tilde{\varphi}}_1(t) \right](\tau) d\tau \]
\[ = \int_{\varepsilon - \theta}^0 \int_{\max(0, t - \varepsilon - \tau)}^t \frac{\partial}{\partial \tau} M_1(s, s - t +\tau)\tilde{V}^{-1}d\varepsilon(s) \]
\[ = \int_{\max(0, t - \varepsilon - \theta)}^t \frac{\partial}{\partial \tau} M_1(s, s - t +\tau)\tilde{V}^{-1}d\varepsilon(s) \]
\[ = \int_{\max(0, t - \varepsilon - \theta)}^t M_1(s, s - t +\theta)\tilde{V}^{-1}d\varepsilon(s) \]
\[ = \left[ \int_0^t T_1(t - s)M_1(s, \cdot)\tilde{V}^{-1}d\varepsilon(s) \right](\theta), \]

i.e. \( \tilde{\varphi}_1 \) is a mild solution of the equation (29). To show that \( \tilde{\varphi}_1 \) is an ordinary solution of the equation (29), define the graph norm in \( D(-d/d\theta) \) by
\[ \| f \|_{D(-d/d\theta)}^2 = \| f \|_X^2 + \| f' \|_X^2. \]

Then \( D(-d/d\theta) \) becomes a Hilbert space and
\[ -d/d\theta \in \mathcal{L}(D(-d/d\theta), \tilde{X}). \]

Using
\[ M_1(t, -\varepsilon) = P_{01}^*(t, -\varepsilon)C^* + P_{12}(t, -\varepsilon, 0) = 0, \quad t \geq 0, \]
one can easily verify that the restriction of \( M_1(\cdot, \cdot)h \) to the set \([0, t] \times [-\varepsilon, 0] \) belongs to \( L_\infty(0, t; D(-d/d\theta)) \) for all \( t > 0 \) and for all \( h \in \mathbb{Z} \). This suffices to conclude that \( \tilde{\varphi}_1 \) is an ordinary solution of the equation (29). In a similar manner, it can be proved that \( \tilde{\varphi}_2 \) is a mild solution of the equation (30).

6 Conclusions

In this paper an approach to wide-band noise, based on a distributed delay of white noise, is developed. This approach allows to reduce the state-observation system disturbed by both white and wide-band noise processes to a linear system, disturbed by only white noise. This reduction is applied to linear filtering and the complete set of formulae for the optimal filter is obtained when both the linear state system and the linear observation system are perturbed by the sum of white and wide-band noise processes.

The three major steps are needed to realize the formulae for the optimal filter:

(a) Computation of the relaxing functions \( \Phi_1 \) and \( \Phi_2 \) under given autocovariance functions \( \Lambda_1 \) and \( \Lambda_2 \) of the wide-band noise processes \( \varphi_1 \) and \( \varphi_2 \) acting to the state and observation systems. This can be done by use of the direct and inverse Fourier transformations as it is described in Section 2.

(b) Numerical solution of the Riccati equation (19) and the system of equations (20)–(21) and (23)–(25) under initial and boundary conditions in (22) and (26).

(c) Design the optimal filter in the linear feedback form by use of the formulae (27) and (28)–(31).

References


