On Riccati equations, orthogonal polynomials and fast filtering algorithm

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Abstract

In this note we cast parametrization of certain type of covariance extension problems in polynomial model. We show that the Kimura form of state space realization for maximum entropy solution to Carathéodory’s extension problem is Riccati balanced parametrization. As an application we provide an extension to Carathéodory’s extension problem is Riccati balanced form. As an application we provide an extension to Carathéodory’s extension problem is Riccati balanced parametrization. As an application we provide a solution to the (CEP). Toeplitz proved that a solution to (CEP) exists, if and only if

\[ T_n := \begin{bmatrix} 1 & c_1 & \cdots & c_n \\ c_1 & 1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & 1 \end{bmatrix} > 0. \] (1)

A recursive solution of the extension problem was constructed by Schur [17].

Each solution \( v(z) \) to the (CEP) gives rise to a spectral factor \( w(z) \) which satisfies

\[ v(z) + v(z^{-1}) = w(z)w(z^{-1}). \]

The linear system with transfer function \( w(z) \) is called a modelling filter for the covariance sequence \( C_n \). The most natural and useful class of solutions to (CEP) in systems theory and applications maybe is the class of rational functions of degree \( n \). Solutions in this class generate modelling filters of order \( n \). Under positive definite condition of the Toeplitz matrix, there exist infinitely many solutions in this class. In this note we study the balanced canonical form of the so-called maximum entropy extension, a solution to the (CEP).

The rest of this note is organized as follows. In Section 2 we review some parametrization by orthogonal polynomial using polynomial models and thereafter in Section 3 we reproduce Kimura’s realization, using the polynomial model. We shall show, in Section 4, that the Kimura realization to the maximum entropy extension is Riccati balanced canonical form. As an application we give an model reduction of covariance extension problem in Section 6. Finally we conclude this note by a sequence of further questions and remarks.

2 Preliminaries

In this section we will give some necessary notations, properties of orthogonal polynomials and positive real functions. The results in this section are after Geronimus [8]. Here we intend to express these results in system-theoretic terms, show how part of the proofs simplify for underlying rational functions, using polynomial models, see Fuhrmann [6] for details.

2.1 Parametrization by orthogonal polynomials and Schur parameters

Let \( \mathbb{R}[z] \) be a set of polynomials of real coefficients. The matrix \( T_n \) is nonnegative definite if and only if there is a positive measure \( d\omega \) defined on the interval \([0, 2\pi]\), that admits the numbers \( C_n \) as its first trigonometric moments:

\[ c_k = \int_0^{2\pi} e^{-ik\theta} d\omega(\theta), \quad k = 0, 1, 2, \ldots, n. \]

With this measure we can define a positive definite inner
product on \( \mathbb{R}[z] \):
\[
\langle a, b \rangle_C := \int_0^{2\pi} a(e^{i\theta})\overline{b(e^{i\theta})}d\omega(\theta), \quad \forall a, b \in \mathbb{R}[z]
\]
We can rewrite the inner product by an infinite Toeplitz matrix associated with the positive sequence \( C := \{\frac{1}{2}, c_1, c_2, \ldots\} \), that is \( v(z) \) with these coefficients in expansion is positive real,
\[
\begin{bmatrix}
a_0 & a_1 & \cdots & a_n \\
b_0 & b_1 & \cdots & b_n \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}T_n
\]

where a polynomial is identified with its vector of coefficients and \( n \) is chosen such that
\[
n \geq \max(\deg a, \deg b)
\]
The inner product defined above has the important property:
\[
\langle za, zb \rangle_C = \langle a, b \rangle_C
\] (2)
The polynomials \( a(z), b(z) \) are said to be orthogonal on the unit circle if \( \langle a, b \rangle_C = 0 \).

We now apply the Gram-Schmidt orthogonalization procedure to the polynomials \( 1, z, z^2, \ldots \) to obtain a sequence of polynomials \( \varphi_n \), orthogonal with respect to the inner product introduced above. Instead of using the normalization \( \| \varphi_n \|_C = 1 \), we require all the \( \varphi_n \) to be monic.

For a polynomial \( p_n(z) \in \mathbb{R}[z] \) of degree \( n \), the reversed polynomial \( p_n^\sharp(z) \) is defined by
\[
p_n^\sharp(z) := z^n p_n(z^{-1}).
\]

Actually we will usually consider the reversion of polynomials as a linear operation in the polynomial model \( X_{\varphi_{n+1}} \), consisting of all polynomials whose degree is at most \( n \). We apply the reversion to all polynomials in \( X_{\varphi_{n+1}} \) as if they had degree \( n \), that is, we define \( p^\sharp(z) = z^n p(z^{-1}) \) for all \( p \in X_{\varphi_{n+1}} \). With this definition it follows from the symmetry of the Toeplitz matrix that, given \( p, q \in X_{\varphi_{n+1}} \), we have
\[
\langle p, q \rangle_C = \langle p^\sharp, q^\sharp \rangle_C
\] (3)

Now we summarize some propositions of orthogonal polynomials \( \varphi_n(z) \) without proof.

**Proposition 2.1.** Let \( C \) be a positive sequence, and let \( \varphi_i \) be the orthogonal polynomials associated with the Toeplitz matrix in (1).

(i) We have
\[
\langle \varphi_n^\sharp, z^n-i \rangle_C = 0, \text{ for } i = 0, \ldots, n-1
\]
(ii) The reverse polynomial \( \varphi_n^\sharp \) are given by
\[
\varphi_n^\sharp(z) = 1 - z \sum_{i=0}^{n-1} \gamma_{i+1} \varphi_i(z),
\]
where the \( \gamma_i \) are defined by
\[
\gamma_{i+1} = \frac{\langle 1, z \varphi_i \rangle_C}{\| \varphi_i \|_C^2}
\]
The coefficients \( \gamma_i \) are called the Schur parameters of the sequence \( C \).

(iii) The orthogonal polynomials satisfy the following recursions
\[
\begin{cases}
\varphi_n = z \varphi_{n-1} - \gamma_n \varphi_{n-1} \\
\varphi_n^\sharp = \varphi_{n-1} - z \gamma_n \varphi_{n-1}
\end{cases}
\] (4)
with the initial conditions \( \varphi_0(z) = \varphi_0^\sharp(z) = 1 \). Equivalently
\[
\begin{bmatrix}
\varphi_n \\
\varphi_n^\sharp
\end{bmatrix} =
\begin{bmatrix}
z & -\gamma_n \\
-\gamma_n z & 1
\end{bmatrix}
\begin{bmatrix}
\varphi_{n-1} \\
\varphi_{n-1}^\sharp
\end{bmatrix},
\begin{bmatrix}
\varphi_0 \\
\varphi_0^\sharp
\end{bmatrix} =
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\] (5)
(iv) We have
\[
\gamma_n = -\varphi_n(0).
\] (6)
(v) The sequence \( \{\| \varphi_n \|_C^2\} \) satisfies the following recursion:
\[
\| \varphi_n \|_C^2 = (1 - \gamma_n^2) \| \varphi_{n-1} \|_C^2, \| \varphi_0 \|_C^2 = 1
\] (7)
Thus we have \( \| \varphi_n \|_C^2 \prod_{j=1}^{n} (1 - \gamma_j^2) \) and
\[
\langle 1, z \varphi_i \rangle_C = \gamma_{i+1} \prod_{j=1}^{i} (1 - \gamma_j^2).
\] (8)
(vi) The Schur parameters \( \gamma_n \) satisfy
\[
| \gamma_n | < 1
\]
for \( n \geq 1 \). In accordance with (6), we define \( \gamma_0 := -1 \).

A direct consequence of this proposition is the celebrated Levinson algorithm for fast computation of the orthogonal polynomials, directly from the data.

**Theorem 2.2.** Let \( \varphi_n(z) = z^n + \sum_{i=0}^{n-1} \varphi_n^i z^i \) be the monic orthogonal polynomials associated with the sequence \( C_n \). Then they can be computed recursively by (we set \( \varphi_{n,n} = 1 \))
\[
r_{n+1} = (1 - \gamma_{n+1}^2) r_n, \quad r_0 = 1;
\]
\[
\gamma_{n+1} = \frac{1}{\gamma_n} \sum_{i=0}^{n} c_{n+i} \varphi_{n,i}, \quad \gamma_0 = -1;
\]
\[
\varphi_{n+1}(z) = z \varphi_n(z) - \gamma_{n+1} \varphi_n^\sharp(z), \quad \varphi_0(z) = 1.
\] (9)

### 2.2 Positive real functions

We recall that a function \( f \) is positive real in the exterior of the unit disc if it is analytic there and its real part is positive. Such functions have a power series expansion of the form
\[
v(z) = 1 + 2 \sum_{k=1}^{\infty} \frac{c_k}{z^k}
\]
The following famous result of Schur characterizes positive real functions:
Theorem 2.3. A function \( v(z) = 1 + 2 \sum_{k=1}^{\infty} c_k z^k \) with \( c_k \in \mathbb{R} \) is a positive real function if and only if the sequence \( C \) is positive sequence or for some \( n \geq 1 \),

\[ \det T_k > 0 \text{ for } 0 \leq k < n \text{ and } \det T_k = 0 \text{ for } k \geq n. \]

The ring of formal power series in the variable \( z^{-1} \) is denoted by

\[ \mathbb{R}[[z^{-1}]] = \left\{ \sum_{k=0}^{\infty} a_k z^{-k} | a_k \in \mathbb{R} \right\} \]

and \( \mathbb{R}_-[[z^{-1}]] \) denotes the subset of power series with \( a_0 = 0 \). Similarly, the ring of formal Laurent series in \( z^{-1} \) is

\[ \mathbb{R}((z^{-1})) = \{ \sum_{k=k_0}^{\infty} a_k z^{-k} | a_k \in \mathbb{R}, k_0 \in \mathbb{Z} \}. \]

The direct sum decomposition

\[ \mathbb{R}((z^{-1})) = \mathbb{R}[z] \oplus \mathbb{R}_-[[z^{-1}]] \]

gives rise to a projection operator \( \pi_+ : \mathbb{R}((z^{-1})) \to \mathbb{R}[z] \).

We will need in the sequel the concept of a partial realization.

Definition 2.4. Suppose that \( g \) is analytic in a neighborhood of infinity, i.e., it has a power series representation of the form \( g(z) = \sum_{k=0}^{\infty} g_k z^{-k} \). We say that a pair of coprime polynomials \( \pi, \chi \) are a partial realization of \( g \) of order \( n \) if

\[ \frac{v(z)}{\chi(z)} = \sum_{k=n+1}^{\infty} \frac{d_k}{z^k}. \]

Then we can write

\[ v \chi - \pi = \chi z^{-(n+1)} d \]

with \( d \in \mathbb{R}[[z^{-1}]] \). This implies that

\[ z^{n-\deg \chi} (v \chi - \pi) = z^{-\deg \chi} \chi d \in \mathbb{R}_-[[z^{-1}]] \]

which can be rewritten as \( \pi_+ z^{n-\deg \chi} (v \chi - \pi) = 0 \), or equivalently, as

\[ \pi(z) = z^{-n+\deg \chi} \pi_+ z^{n-\deg \chi} (v \chi(z)). \]

Given a positive real function \( v(z) = \frac{1}{2} + \sum_{k=1}^{\infty} c_k z^k \), define the inner product on \( \mathbb{R}[z] \) that corresponds to the sequence \( C = \{\frac{1}{2}, c_1, c_2, \ldots\} \) of its coefficients, and the associated orthogonal polynomials \( \varphi_n \). We define now a sequence of polynomials \( \psi_n \) by

\[ \psi_n = 2\pi_+ v \varphi_n = 2T_v \varphi_n. \]

Here \( T_v : \mathbb{R}[z] \to \mathbb{R}[z] \) is the Toeplitz map defined by

\[ T_v f = \pi_+ v f. \]

Clearly, by our normalization and the monicity of the \( \varphi_n \), the polynomials \( \psi_n \) are monic. Thus \( \psi_n \) is a partial realization of \( v \) of order \( n \). If \( n \) is the least integer for which \( \det (T_n) = 0 \), then \( \psi_n = 2v \varphi_n \), or

\[ v(z) = \frac{1}{2} \psi_n(z). \]

Since the inverse of a positive real function is also positive real, we conclude that the infinite positive sequence \( \{\frac{1}{2}, c_1, c_2, \ldots\} \) defines another positive sequence \( \{\hat{c}_0, \hat{c}_1, \ldots\} \) which is obtained from

\[ v \hat{\nu} = 1 + \sum_{k=1}^{\infty} \hat{c}_k z^k. \]

In the following, let \( C \) be a positive sequence, let \( v(z) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{c_k}{z^k} \) be the corresponding positive real function with Toeplitz operator \( T_v \). We define another inner product on \( \mathbb{R}[z] \) using the positive sequence \( \hat{C} = \{\hat{c}_0, \hat{c}_1, \ldots\} \) that belongs to \( \hat{v} = 1/v \), i.e.,

\[ \langle p, q \rangle_{\hat{C}} = \sum_{i=1}^{n} \sum_{j=0}^{n} \hat{c}_{i-j} p_i q_j. \]

Then we have:

Proposition 2.5.  (i) For \( p, q \in \mathbb{R}[z] \)

\[ \langle p, q \rangle_{\hat{C}} = \langle T_v p, T_v q \rangle_{\hat{C}}. \]

(ii) If \( \varphi_n \) are the orthogonal polynomials associated with the sequence \( C \), then \( \psi_n \) are the orthogonal polynomials associated with the sequence \( \hat{C} \).

(iii) The Schur parameters for the orthogonal polynomials \( \psi_n \) are given by \(-\gamma_n\).

(iv) The polynomials \( \psi_n \) satisfy the following recursions

\[ \psi_n = z \psi_{n-1} + \gamma_n \psi_{n-1}^d \]

\[ \psi_n^d = \psi_{n-1}^d + z \gamma_n \psi_{n-1}. \]

The initial conditions for the recursions are \( \psi_0(z) = \psi_0^d(z) = 1 \).

3 The Kimura realization of the maximum entropy extension

In this section we will show that the Kimura realization of the maximum entropy extension is in fact a canonical form of the balanced parametrization.

3.1 Derivation of a realization to the maximum entropy extension

First we will derive a realization representation of the maximum entropy extension we discussed in Section 2. Recall
that the polynomial model \( X_{\phi_n} \subset \mathbb{R}[z] \), where \( \phi_n \) is a polynomial of degree \( n \), consists of all polynomials whose degree is less than \( n \). The shift operator in \( X_{\phi_n} \), is the linear transformation \( S_{\phi_n} : X_{\phi_n} \rightarrow X_{\phi_n}, \ p \mapsto \pi_{\phi_n} z p \), where for a polynomial \( f, \pi_{\phi_n} f \) denotes the remainder of \( f \) after division by \( \phi_n \).

The Kimura realization of the maximum entropy extension will become transparent if we use the polynomial model with the normalized orthogonal polynomials as basis. To this end, we need a few more propositions regarding the polynomials \( \varphi_n, \psi_n \).

**Proposition 3.1.** Let \( \varphi_n \) and \( \psi_n \) be the orthogonal polynomials associated with \( C \) and \( \hat{C} \) as in Section. Then we have, for \( n \geq 1 \)

\[
\varphi_n^2 + \psi_n^2 = 2z^n \prod_{j=1}^{n} (1 - \gamma_j^2) = 2z^n \| \varphi_n \|_{\hat{C}}^2.
\]

**Proposition 3.2.** Let \( \varphi_i \) be the orthogonal polynomials associated with a positive sequence \( C \). We have

\[
z \varphi_i = \varphi_{i+1} - \gamma_{i+1} r_i \sum_{j=0}^{i} \frac{\gamma_j}{r_j} \varphi_j, \quad i \leq n - 2
\]

and

\[
z \varphi_{n-1} - \varphi_n = -\gamma_n r_{n-1} \sum_{j=0}^{n-1} \frac{\gamma_j}{r_j} \varphi_j,
\]

where \( \gamma_i \) are the Schur parameters of \( C \) (with \( \gamma_0 = -1 \)) and \( r_i \) are the parameters of the Levinson algorithm (9).

**Proof.** It suffices to show that for all \( i, \)

\[
z \varphi_i = \varphi_{i+1} - \gamma_{i+1} r_i \sum_{j=0}^{i} \frac{\gamma_j}{r_j} \varphi_j.
\]

Then both (15) and (16) follow. In view of the recurrence relation (4), that is,

\[
z \varphi_i = \varphi_{i+1} + \gamma_{i+1} \varphi_i^d,
\]

we need to prove that

\[
\varphi_i^d = -r_i \sum_{j=0}^{i} \frac{\gamma_j}{r_j} \varphi_j.
\]

This is accomplished by the following lemma.
Proof. The proof is by induction on $k$. For $k = i + 1$ we compute

$$\psi_{i+1}^{\sharp} \varphi_i + \varphi_{i+1} \psi_i = \psi_i \varphi_i + \varphi_i \psi_i$$

From (13) we conclude that $\deg(\psi_{i+1}^{\sharp} \varphi_i + \varphi_{i+1} \psi_i) = i$. Similarly, (14) implies that $\deg(\psi_i \varphi_i + \varphi_i \psi_i) = i$ with leading coefficient $2\gamma_{i+1}r_i$. Assume that we have proved the assertion for the indices $k = i + 1, \ldots, m$. Then

$$\psi_{m+1} \varphi_i + \varphi_{m+1+1} \psi_i = (\psi_m + \gamma_{m+1} \psi_m) \varphi_i + (\varphi_m - \gamma_{m+1} \psi_m) \psi_i$$

This shows that $\deg(\psi_{m+1} \varphi_i + \varphi_{m+1} \psi_i) = m$, the coefficient of $z^m$ being the leading coefficient of the first summand, namely $2\gamma_{i+1}r_i$. 

Now we are in the position to derive the Kimura realization.

**Proposition 3.5.** Given a positive sequence $C$, let $v(z) = \frac{1}{2} + \sum_{i=1}^{\infty} \frac{c_i}{z^i}$ be the corresponding positive real function. Let $\varphi_i, \psi_i$ be the associated orthogonal polynomials of the first and second kind. Then a realization of $\frac{1}{2} \frac{v_n}{\varphi_n}$ is given by

$$F = \begin{bmatrix} \gamma_1 & \gamma_1 \gamma_2 & \gamma_1 \gamma_2 \gamma_3 & \cdots & \gamma_1 \cdots \gamma_{n-1} \gamma_n \\ \gamma_1^2 & -\gamma_1 \gamma_2 & -\gamma_1 \gamma_2 \gamma_3 & \cdots & -\gamma_1 \gamma_2 \cdots \gamma_{n-1} \gamma_n \\ \gamma_2 & -\gamma_2 \gamma_3 & -\gamma_2 \gamma_3 \gamma_4 & \cdots & -\gamma_2 \gamma_3 \cdots \gamma_{n-1} \gamma_n \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \gamma_{n-1} & -\gamma_{n-1} \gamma_n & -\gamma_{n-1} \gamma_n \gamma_{n+1} & \cdots & -\gamma_{n-1} \gamma_n \cdots \gamma_{n+1} \gamma_n \\ \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T = e_1,$$

$$c = \begin{bmatrix} \gamma_1 & \gamma_1 \gamma_2 & \gamma_1 \gamma_2 \gamma_3 & \cdots & \gamma_1 \cdots \gamma_{n-1} \gamma_n \\ \end{bmatrix} = e_1^T F,$$

$$d = \frac{1}{2}$$

where $\gamma_i^2 = \sqrt{1 - \gamma_i^2}$ are called the complementary of $\gamma_i$ and $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Proof. It is clear that $\lim_{z \to \infty} \frac{1}{2} \frac{\psi_n(z)}{\varphi_n(z)} = \frac{1}{2}$, hence $d = \frac{1}{2}$ and the problem reduces to finding a realization of the strictly proper function

$$\frac{1}{2} \frac{\psi_n}{\varphi_n} = \frac{1}{2} \frac{\psi_n - \varphi_n}{\varphi_n}.$$
The items (i), (ii) and (iv)-(vi) are from Kimura [9], while (iii) can be proven by a standard matrix factorization technique used in numerical analysis. From Proposition 3.6 (iii) we have the following

Corollary 3.7. The realization \((F, b, c, d)\) of the maximum entropy extension is minimal if and only if \(\gamma_n \neq 0\).

From now on we will assume that \(\gamma_n \neq 0\) without loss of generality, because otherwise we may choose a smaller \(n\) for which this condition holds. In fact, \(\gamma_n = -\varphi_n(0)\) then \(\varphi_n(z) = z\varphi_{n-1}(z)\) by (4) and similar expression holds for \(\psi_n(z)\). Hence we can cancel the factor \(z\).

3.2 Balanced parametrization

Now we will show that the Kimura realization in Proposition 3.5 is a Riccati balanced canonical form. To this end we need two special modelling filters. See Lindquist and Picci [11, 12, 13] for more details and relations to stochastic real-

We know that \(\frac{1}{2} \varphi_n^T\varphi_n\) is positive real. Hence the following linear matrix inequality (LMI) is solvable:

\[
M(P) = \begin{bmatrix} P - FP F^T & (I - FP F^T)e_1 \\ (I - FP F^T) e_1^T & e_1^T (I - FP F^T) e_1 \end{bmatrix} \geq 0.
\]

By Lemma 3.6 (ii) and (iv) it is obvious that \(P = I\) is a solution to (23). This gives us the factorization of the matrix \(M(P)\):

\[
M(P) = \begin{bmatrix} \delta_n F e_n \sqrt{\tau_n} \\ \sqrt{\tau_n} \end{bmatrix} \begin{bmatrix} \delta_n e_n^T F^T \\ \sqrt{\tau_n} \end{bmatrix} \]

Kimura showed that [9]

Lemma 3.8. The minimal solution to the LMI (23) is \(P_+ = I\).

The maximum entropy extension to (CEP) gives rise the minimum phase spectral factorization of the spectral density

\[
\frac{r_n}{\varphi_n(z)\varphi(z^{-1})} = w_-(z)w_-(z^{-1})
\]

where

\[
w_-(z) = \frac{\sqrt{\tau_n} e_n}{\varphi_n(z)}.
\]

This is also called the maximum entropy solution to the spectral factorization. Combining (24) and Lemma 3.8 yields a realization of \(w_-(z)\),

\[
w_-(z) = \sqrt{\tau_n} + e_n^T F(zI - F)^{-1}(\delta_n F e_n).
\]

Obviously

\[
w_+(z) = \frac{\sqrt{\tau_n}}{\varphi_n(z)}
\]

is also a spectral factor of the same spectral density but has all zeros outside the unit circle. This is called the maximum phase spectral factor in filtering theory. One of its realizations is readily obtained using Proposition 3.6 (vi), that is,

\[
w_+(z) = \frac{\gamma_n^T e_n^T (zI - F)^T e_1}{\sqrt{\tau_n}}
\]

Let us now recall definitions of balanced realizations and canonical forms of realizations. Denote a realization of a rational function \(G(z) := \begin{bmatrix} A & B \\ C & D \end{bmatrix}\). The following definitions are taken from Ober [15].

Definition 3.9. Let \(M\) be a set of minimal state-space systems. Then a map \(\Gamma : M \rightarrow M\) is called a canonical form if

(i) \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is a solution due to Proposition 3.6 (iii). Hence there is \(T \in \mathbb{R}^{n \times n}\) invertible such that

\(\hat{A} = TAT^{-1}, \hat{B} = T B, \hat{C} = CT^{-1}, \hat{D} = D.\)

(ii) If \(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\) and \(\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\) are system equivalent then

\(\Gamma \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \right) = \Gamma \left( \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \right).\)

Definition 3.10. A minimal realization \(\begin{bmatrix} F & b \\ c & d \end{bmatrix}\) is called Riccati balanced if the minimal solution to the positive real algebraic Riccati equation (PRARE)

\[
P = FP F^T + (b - FP c^T)(1 - c Pc^T)^{-1}(b - FP c^T)^T
\]

and the minimal solution to the dual (DPRARE)

\[
P = F^T P F + (c^T - TF^T P b)(1 - b^T P b)^{-1}(c^T - TF^T P b)^T
\]

diagonal and equal.

Note that the positive real Riccati equation and its dual are written in our precise meaning.

Theorem 3.11. Assume that \(\gamma_n \neq 0\), that is, the realization \(\begin{bmatrix} F & b \\ c & d \end{bmatrix}\) obtained in Proposition 3.5 is minimal. Then, it is Riccati balanced canonical form.

Proof. By Lemma 3.8 we conclude that the minimal solution of the (PRARE) which corresponds the minimum phase spectral factor. It is known that the minimal solution of (DPRARE) is the minimal solution of the following LMI

\[
M(X) = \begin{bmatrix} X - F^T X F & F^T (I - X) e_1 \\ (F^T (I - X) e_1)^T & e_1^T (I - X) e_1 \end{bmatrix} \geq 0.
\]

It is clear that \(X = I\) is a solution due to Proposition 3.6 (iii). Again using the same property we obtain

\[
M(X) = \begin{bmatrix} \gamma_n^T e_n & 0 \\ 0 & \gamma_n^T e_n \end{bmatrix}
\]
giving the realization of the maximum phase spectral factor. Hence the solution \( X = I \) is indeed the minimal solution of the (DPRARE). So the the Kimura realization is Riccati balanced.

4 Model reduction technique

One of the main advantages of balanced representations is that they can be used for a very straightforward method of model reduction, see e.g. Moore [14], Pernebo and Silverman [16] etc. Their result assumes that truncation does not occur between states corresponding to repeated singular values. Otherwise the approximation may no longer asymptotically stable and minimal. Ober [15] however suggested a model reduction technique called parameter projection method that performs the model reduction by reducing the parameters. A reduced-order system of degree, say \( N \) can easily be defined by retaining of the parameters only those that corresponding to the first \( N \) states. It is shown that these parameters define a unique reduced-order system that is in the same class of the systems as the original system.

Since our parametrization of the maximum entropy extension is in the same setting of that of Ober’s. We can use Ober’s parameter projection method to reduce the order of a system in a desired class, in our case, to keep systems positive real. We can simply discard the last Schur parameters to reduce the order of the systems, and the principal subsystem is still positive real. In fact the reduced modelling filter is still of maximum entropy solution. Discarding last Schur parameters is a good approximation using a result of the so-called geometric convergence from Byrnes, Lindquist, Guseev and Matveev [1]

**Lemma 4.1.** Let the spectral density \( \Phi(e^{i\theta}) = \|W(e^{i\theta})\| \) be coercive in the sense that it is positive for all \( \theta \) and let \( \gamma_1, \gamma_2, \ldots \) be the corresponding infinite sequence of Schur parameters. Moreover, let \( \gamma \in (0, 1) \) be greater than the maximum of moduli of the zeros of \( W(z) \). Then

\[ |\gamma_t| = O(\gamma^t) \]

i.e., \( |\gamma_t| \leq M\gamma^t \) for some real number \( M \) and for sufficiently large \( t \).

In this note we want to carry the problem further. Is it possible to find an approximation of a positive real function which matches the given partial sequence \( C_n \) with both poles and zeros in the modelling filter? To get an answer to this question we need some details concerning Kimura-Georgiou parametrization and some new results from Byrnes et al.

As mentioned in the beginning we want to find a rational extension to a given partial sequence containing the first \( n+1 \) coefficients in expansion of a positive real rational function. Intuitively, solutions in the class of rational functions of degree \( n \) contain \( n \) free parameters, because a rational function of degree \( n \) possesses \( 2n + 1 \) coefficients and the number of constraints imposed by data is \( n + 1 \). To cope with these free parameters, Kimura [9] and Georgiou [7] introduced an alternative way to describe a positive real function \( (z) \) in terms of the first Schur parameters \( \gamma_1, \gamma_2, \ldots, \gamma_n \), namely

\[
v(z) = \frac{1}{2} \psi_n(z) + \frac{\alpha_1}{2} \psi_{n-1}(z) + \cdots + \frac{\alpha_n}{2} \psi_0(z)
\]

where \( \alpha_1, \ldots, \alpha_n \) are real numbers. Nevertheless, this parametrization holds regardless of positivity. In fact it was shown [3] that it defines a birational isomorphism \( \mathbb{R}^{2n} \), i.e. it defines a birational change of coordinates.

In [3] it is also established that the window of Schur parameters \( (\gamma_{t+1}, \ldots, \gamma_{t+n})^T \) is generated by the dynamical system, a reformulation of the fast filtering algorithm due to Lindquist [10]

\[
\alpha(t+1) = A(\gamma(t))\alpha(t), \quad \alpha(0) = \alpha \quad (28)
\]

\[
\gamma(t+1) = G(\alpha(t+1))\gamma(t), \quad \gamma(0) = \gamma \quad (29)
\]

where the maps \( A, G : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n} \) are defined as

\[
A(\gamma) = \begin{bmatrix}
\frac{1}{1-\gamma_1^2} & \frac{\gamma_1 \gamma_{n-1}^2}{(1-\gamma_2^2)(1-\gamma_{n-1}^2)} & \cdots & \frac{\gamma_n \gamma_1^2}{(1-\gamma_{n-2}^2)(1-\gamma_{n-1}^2)} \\
0 & \frac{1}{1-\gamma_2^2} & \cdots & \frac{\gamma_n \gamma_{1}^2}{(1-\gamma_{n-2}^2)(1-\gamma_{n-1}^2)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{1-\gamma_n^2}
\end{bmatrix},
\]

\[
G(\alpha) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1
\end{bmatrix}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \) and \( \gamma = (\gamma_1, \ldots, \gamma_n)^T \).

If \( v(z) = \frac{1}{2} \psi_n(z) \) is a positive real function, not only do the Schur parameters satisfy \( |\gamma_t| < 1 \) for all \( t \), but they also converge to zero, geometrically. Moreover, the polynomial

\[
\alpha_{\infty}(z) = z^n + \alpha_{\infty}z^{n-1} + \cdots + \alpha_{\infty
}
\]

is a Schur polynomial. Then the minimum phase spectral factor of \( \phi(z) = v(z) + v(z^{-1}) \) is

\[
w(z) = \sqrt{\alpha_{\infty}(z)}.
\]

This shows that the above nonlinear dynamical system is an iterative algorithm to find the minimum phase spectral factor if the parameters \( \alpha_i \) correspond to a positive real function. It also shows clearly that the Schur parameters can be obtained by linear combination of the anew computed \( \alpha \) and previous \( n \) Schur parameters. That is, we only need \( n \) Schur parameters in every iteration, once we have the right parametrization \( \alpha \) for positive real function \( \gamma \). The latter is however not a trivial problem.

Now we turn to the following problem: Given \( C_n = \{ \frac{1}{2}, c_1, \ldots, c_n \} \). Find an approximate extension \( \tilde{v}(z) \) of a positive real rational function \( v(z) \) such that its first coefficients
match the given data $C_n$ and the resulting modelling filter has both poles and zeros (for some purpose it is desired to have zeros in the transfer function, see [2] for motivations).

By Theorem 3.11 we know that an order reduction can be done by setting the last Schur parameters to 0. Lemma 4.1 says that if $n$ is large enough $\gamma_n = O(\gamma^n)$ with $|\gamma| < 1$. Now assume that we have the reduced order $N$ such that $\frac{n}{2} \leq N < n$ then it is possible to find $N$ constants $\alpha_1, \ldots, \alpha_N$ such that the nonlinear systems (28) and (29) converge to a limit which gives us a Schur polynomial $\alpha_\infty(z)$. This argument is based on the following theorem by [2].

**Theorem 4.2.** Let $n^*$ be any integer satisfying $0 \leq n^* \leq n$. Then the subset $S(n^*)$ of $\mathbb{R}^n$, consisting of partial covariance sequences $C_n$ having a minimal stochastic realization of degree $n^*$, are a nonempty semialgebraic set. The subset $\Sigma(n^*)$ of those partial covariance $C_n$ having a minimal stochastic realization of degree less than or equal to $n^*$ is also semialgebraic. Moreover, $S(n^*)$ and $\Sigma(n^*)$ have nonempty interiors if and only if $\frac{1}{n^*} \leq \frac{1}{n}$. In theory, we obtain the maximum entropy solution $\hat{\alpha}^*$.

**Remark 4.3.** Equation (30) may fail to have solutions in general. On the other hand, we know that $\alpha_1 = \cdots = \alpha_N = 0$ is always a solution, because it corresponds to the maximum entropy solution. If we start the initial guess $\alpha_1 = \cdots = \alpha_N = 0$ in an optimization algorithm to solve (30) we may be able to find a local optimum, if there is one. We can also replace $N$ parameters by $n - N$ parameters. According to the structure of the fast filtering algorithm (28) and (29) we can obtain $n - N$ parameters $\alpha_1, \ldots, \alpha_{n-N}$ directly, using exactly so many Schur parameters required in (30). In theory, such $n - N$ parameters $\alpha_1, \ldots, \alpha_{n-N}$ exist.

5 Conclusion

We discussed the canonical form of a special class of positive real function in this note. We proved that the Kimura realization of the maximum extension technique is Riccati balanced. This leads to a natural model reduction technique of Ober’s. This parametrization has advantage in model reduction in the sense that we can reduce systems order in a desired class without restriction on truncation states corresponding to repeated singular values. As an application we proposed an algorithm for finding pole-zero modelling filter that matches the first given data based on the argument of model reduction. We believe that this technique can be used in other contexts, for example, identification methods. The most natural questions concerning our approach is how we find a balanced canonical form for a more general class of positive real function and if it can be carried over to multivariable case. We have partial answer to the second question which will be studied further. However, we are pessimistic about an algebraic approach to the first one at this moment.

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References


