Stability analysis of one and two-dimensional continuous systems with parameters

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Keywords: stability analysis, two dimensional systems, optimal control system.

Abstract
In the paper one and two-dimensional optimal control systems with perturbations are considered. Some sufficient conditions under which optimal processes continuously (or semicontinuously) depend on variable parameters of the system are proved.

1 Introduction
In the theories of boundary value problems, initial value or optimization problems the most important is the question of the existence of a solution. When we know that the solution does exist, we consider the question of the uniqueness and dependence on boundary data and parameters of the system considered.

Following Hadamard, we say that a given initial value, boundary value, optimal control problem etc. is well-posed if this problem

a) possesses a solution,

b) the solution is unique,

c) the solution continuously depends on parameters of the system.

The ill-posedness of a problem is usually related to condition c). In general we say that a given problem is ill-posed if its solution exists, but it is not a continuous function of parameters or boundary data.

Hadamard found a simple example of an ill-posed initial value problem for P.D.E.

Example 1 Consider the Laplace equation

\[
\begin{align*}
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= 0, \\
(x,y) &\in P = \{(x,y) \in \mathbb{R}^2 : x \in [a,b], y \in [0,c]\}
\end{align*}
\]  

with the initial conditions

\[
\begin{align*}
z(x,0) &= \varphi_n(x) = \frac{1}{n^2} \sin(nx), \\
\frac{\partial z}{\partial y}(x,0) &= \psi_n(x) = \frac{1}{n} \sin(nx).
\end{align*}
\]  

By a direct inspection and Carleman’s theorem we can show that the function

\[ z_n(x,y) = \exp(ny) \cdot \sin(nx) \]

is the unique solution of problem (1)-(2) in the space \( C^2(P, \mathbb{R}) \) for \( n = 1, 2, \ldots \).

Passing with \( n \) to infinity, we see that \( \varphi_n(\cdot) \) and \( \psi_n(\cdot) \) tend to null uniformly (in \( C^0([a,b], \mathbb{R}) \)) but the sequence \( z_n(\cdot, \cdot) \) does not converge to the function \( z_0(x,y) \equiv 0 \) which is the unique solution of the Laplace equation with homogeneous initial data. Thus problem (1)-(2) is ill-posed.

Next we give an example of an ill-posed optimal control problem.

Example 2 Let us consider a two dimensional system with a scalar parameter \( w \in [0,1] \)

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u - wx_1(t), \\
u &\in [-1,1],
\end{align*}
\]  

with the boundary conditions

\[
\begin{align*}
x_1(0) &= 0, \\
x_1(\pi) &= 0, \\
x_2(0) \text{ and } x_2(\pi) \text{ are free}
\end{align*}
\]  

and with the cost functional

\[
J_w(x,u) = \int_0^\pi x_1(t) \left( x_1(t) - 10^3 \sqrt{2\pi} \right) dt \rightarrow \inf.
\]

Optimal control system (3)-(5) is a linear system with quadratic cost functional, thus it is easy to show that for any parameter \( w \in [0,1] \) this problem possesses an optimal solution \( (x_w^*, u_w^*) \).

By a direct estimation, we can show that for any \( w \in (0,1) \)

\[
J_w(x_w^*, u_w^*) > -4 \cdot 10^4.
\]
For $w = 1$, control system (3)-(4) possesses one optimal process $u_1^* = 0$ and
\[ x_1^*(t) = \sqrt{\frac{2}{\pi}} 10^3 (\sin t, \cos t). \]

By a direct calculation one can show that
\[ J_1 = -10^6. \tag{7} \]

If we compare (6) and (7), we see that by passing with $w \to 1$, performance index (5) has a big jump, thus it is not continuous at the point $w = 1$ and the optimal control problem (3)-(5) under consideration is not well-posed (is ill-posed).

In the last years, stability analysis for various optimization and control problems was considered in many papers and monographs (cf. survey paper [1] and references therein).

In this paper we give a direct method of stability analysis for some optimal control systems and we prove sufficient conditions under which the set of optimal processes and the optimal value continuously depend on variations of parameters of the control system.

2 Formulation of the one-dimensional problem. Basic assumptions and lemmas

Let us consider an optimal control system $(P_0)$ described by the nonlinear O.D.E.
\[
\begin{cases}
  \dot{x} = \varphi_1^0(t, x) + \varphi_2^0(t, x) u, \\
  x(0) = p_0 \in \mathbb{R}^n, \\
  u(.) \in U_0 = \{ u(.) \in L^2(I, \mathbb{R}^m) : u(t) \in M_0 \}, \\
  t \in I = [0, T], \ T \text{ is fixed},
\end{cases}
\tag{8}
\]

with the performance index
\[
J_0(x, u) = \int_I f_0(t, x(t), u(t)) dt + l_0(x(T)). \tag{9}
\]

Besides system (8)-(9) we shall consider a disturbed system $(P_k)$.
\[
\begin{cases}
  \dot{x} = \varphi_1^k(t, x) + \varphi_2^k(t, x) u, \\
  x(0) = p_k \in \mathbb{R}^n, \ t \in I, \\
  u(.) \in U_k = \{ u(.) \in L^2(I, \mathbb{R}^m) : u(t) \in M_k \}, \\
  J_k(x, u) = \int_I f_k(t, x(t), u(t)) dt + l_k(x(T)). \tag{10}
\end{cases}
\]

Systems (8)-(11) will be investigated in the Hilbert space $H^1(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^m)$, where $H^1$ is the space of all absolutely continuous functions on $I$ such that $\dot{x}(.) \in L^2(I, \mathbb{R}^n)$. The norm in $H^1$ is given by the formula $\|x\|^2 = \int_I (|x(t)|^2 + |\dot{x}(t)|^2) dt$. Further, we impose the following assumptions on the above systems:

(A1) the functions $\varphi_1^k(.)$ are measurable with respect to $t$ and lipschitzian with respect to $x$, i.e. there exists $L > 0$ such that
\[ |\varphi_1^k(t, x^1) - \varphi_1^k(t, x^2)| \leq L |x^1 - x^2| \]
for $t \in I$ a.e., $x^1, x^2 \in \mathbb{R}^n$, $i=1, 2, k=0, 1, 2, ...$.

(A2) there exists a function $b(.) \in L^2(I, \mathbb{R}^+)$ such that
\[ |\varphi_2^k(t, 0)| < b(t) \]
for $t \in I$ a.e., $i = 1, 2, k = 0, 1, 2, ...$.

(A3) the functions $\varphi_1^k(.)$ are close to $\varphi_0^k(.)$ on bounded sets, i.e. for any bounded set $A \subset \mathbb{R}^n$, there exists a sequence of functions $\beta_k(.) \in L^2(I, \mathbb{R}^+)$ such that
\[ |\varphi_1^k(t, x) - \varphi_0^k(t, x)| \leq \beta_k(t) \]
for $t \in I$ a.e., $x \in A$, $i = 1, 2, k = 0, 1, 2, ...$ and $\beta_k(.) \to 0$ in $L^2(I, \mathbb{R})$ if $k \to \infty$.

(A4) for any bounded set $A \subset \mathbb{R}^n$, there exists a function $\gamma(.) \in L^1(I, \mathbb{R}^+)$ such that
\[ |f_k(t, x, u)| \leq \gamma(t) \]
for $t \in I$ a.e., $x \in A$ and $u \in M_k$, $k = 0, 1, 2, ...$.

(A5) the functions $f_k(.)$, $k = 0, 1, 2, ...$ are measurable with respect to $t \in I$, continuous with respect to $(x, u)$ and convex with respect to $u$,.

(A6) the functions $l_k(.)$ are continuous, $l_k(.)$ tends to $l_0(.)$ uniformly on any bounded set $A \subset \mathbb{R}^n$ and the sets $M_k \subset \mathbb{R}^m$ are convex and compact, $k = 0, 1, 2, ...$.

Based ourselves on assumptions (A1)-(A3) one can prove the following lemmas

Lemma 3 If the sets $M_k$, $k = 0, 1, 2, ...$, are convex and compact and $M_k \to M_0$ in the Hausdorff metric, then for any admissible control $u_0(.) \in U_0$ and any $\delta > 0$, $\varepsilon > 0$, there exists $K > 0$ such that, for all $k > K$, there exists $u_k(.) \in U_k$ ($u_k$ may even be continuous) such that
\[ \max \{ t \in I : |u_k(t) - u_0(t)| > \delta \} < \varepsilon. \tag{12} \]

Conversely, for any $u_k(.) \in U_k$ with $k > K$, there exists $u_0(.) \in U_0$ ($u_0$ may be chosen continuous) such that (12) holds.
Denote by \( X_k \times U_k \) the set of admissible processes for the system \( (P_k) \), \( k = 0, 1, 2, \ldots \), given by (8) or (10), i.e. the set \( X_k \) is defined as the set of all solutions of equations (8) or (10) which correspond to some admissible control \( u_k(.) \) in \( U_k \), \( k = 0, 1, 2, \ldots \).

Let us notice that assumptions (A1) and (A2) guarantee that, for any control \( u_k(.) \) in \( U_k \), there exists a unique solution \( x_k(.) \) in \( H^1(I, \mathbb{R}^n) \) of (8) or (10).

By \( X_k^* \times U_k^* \) we shall denote the set of all optimal processes for the optimal control system \( (P_k) \), \( k = 0, 1, 2, \ldots \). Assumptions (A1)-(A6) and the well-known existence theorem imply that the set \( X_k^* \times U_k^* \) is not empty for \( k = 0, 1, 2, \ldots \) (cf. [2]). Of course, \( X_k^* \times U_k^* \subseteq X_k \times U_k \).

Next, by \( m_k \) let us denote the optimal value for the system \( (P_k) \), i.e.

\[
m_k = J_k(x^*, u^*)
\]

for any \( (x^*, u^*) \in X_k^* \times U_k^* \), \( k = 0, 1, 2, \ldots \).

We have the following

**Lemma 4 (Principal lemma).** If

1. the systems \( (P_k) \) defined by (8) and (10) satisfy conditions (A1)-(A3),

2. the sequence of sets \( M_k \) tends to \( M_0 \) in the Hausdorff metric and the sequence of initial points \( p_k \) tends to \( p_0 \) in \( \mathbb{R}^n \),

3. the sequence of cost functionals \( J_k(x, u) \) tends to \( J_0(x, u) \) uniformly on any bounded set in the space \( H^1(I, \mathbb{R}^n) \times L^2(I, \mathbb{M}) \),

then

(a) there exists a ball \( B(0, \rho) \subset H^1 \) such that \( X_k \subset B(0, \rho) \) for \( k = 0, 1, 2, \ldots \), i.e. there exists \( \rho > 0 \) such that, for any admissible trajectory \( x_k(.) \) of the system \( (P_k) \) (cf. (10)) \( \|x_k\|_{H^1} \leq \rho \).

(b) for any \( \varepsilon > 0 \) and any \( x_0(.) \in X_0 \), there exists \( K > 0 \) such that, for all \( k > K \), there exists \( x_k(.) \in X_k \) such that

\[
\|x_k - x_0\|_C < \varepsilon \quad i.e. \quad \max |x_k(t) - x_0(t)| < \varepsilon
\]

and conversely, for any \( \varepsilon > 0 \), we can find \( K > 0 \) such that, for any \( x_k(.) \in X_k \) with \( k > K \), there exists some \( x_0(.) \in X_0 \) such that \( \|x_k - x_0\|_C < \varepsilon \) where \( x_k(.) \) is a trajectory of the system \( (P_k) \) (cf. (8), (10)) which corresponds to the control \( u_k(.) \) in \( U_k \) satisfying (12) and the initial value \( p_k \) sufficiently close to \( p_0 \).

(c) the sequence of optimal values \( m_k \) defined by (13) tends to the optimal value \( m_0 \).

(d) the weak upper limit of the optimal sets \( X_k^* \times U_k^* \subset H^1 \times L^2 \) is a nonempty set \( \overline{X}_k^* \times \overline{U}_k^* \subset X_k^* \times U_k^* \). In the case when the optimal processes \( (x_k^*, u_k^*) \) are unique, i.e. when the sets \( X_k^* \times U_k^* \) are singletons, we have\n
\[
\lim(x_k^*, u_k^*) = (x_0^*, u_0^*) \quad in \quad the \quad weak \quad topology \quad of \quad H^1 \times L^2.
\]

**Remark 5** Let us recall that the upper limit of the sequence of sets \( V_k \) is defined as the set of all cluster points of sequences \( \{v_k\} \) where \( v_k \in V_k \) and \( V_k, k = 1, 2, \ldots \), are subsets of some metrizable space \( Y \).

**Remark 6** The weak convergence of \( x_k^*(.) \) to \( x_0^*(.) \) in \( H^1 \) implies the strong convergence of \( x_k^*(.) \) to \( x_0^*(.) \) in the space \( C \) (cf. [3]). Thus the sequence of optimal trajectories \( x_k^*(.) \) uniformly converges to some optimal trajectory \( x_0^*(.) \) of the problem \( (P_0) \).

3 The main results

Basing ourselves on the fundamental lemma, we can prove some sufficient conditions under which optimal processes continuously depend on variable parameters of an optimal control system. The following theorems hold

**Theorem 7** If

1. the optimal control systems \( (P_k) \), \( k = 0, 1, 2, \ldots \), are given by (8)-(11) and assumptions (A1)-(A6) hold,

2. the sequence of sets \( M_k \) tends to \( M_0 \) in the Hausdorff metric and the sequence of initial values \( p_k \) tends to \( p_0 \) in \( \mathbb{R}^n \),

3. the integrands \( f_k \) tend to \( f_0 \) in \( L^1 \), i.e. for any bounded set \( A \subset \mathbb{R}^n \) and any bounded set \( \overline{M} \subset \mathbb{R}^n \), there exists a sequence of functions \( \gamma_k(.) \in L^1(I, \mathbb{R}^+) \) such that

\[
|f_k(t, x, u) - f_0(t, x, u)| \leq \gamma_k(t)
\]

for \( t \in I \) a.e. \( k = 1, 2, \ldots \), and for all \( (x, u) \in A \times \overline{M} \). Moreover, we assume that \( \gamma_k(.) \) tends to zero in \( L^1 \),

then

(a) the sequence of optimal values \( m_k = \min J_k(x, u) \) tends to \( m_0 = \min J_0(x, u) \),
Similarly as the previous theorem one can prove

\[ \text{Theorem 8 If} \]

1. \( f_k(t, x, u) = F^1(t, w_k(t), x) + \langle F^2(t, w_k(t), x), u \rangle \), \( w_k(.) \in L^p(I, \mathbb{R}^p) \), \( p \in [1, \infty) \), and \( F^i \) is measurable with respect to \( t \) and continuous with respect to \( (w, x) \), \( i = 1, 2, \)

2. for any bounded set \( A \subset \mathbb{R}^n \), there exists \( C > 0 \) such that
\[
|F^i(t, w, x)| \leq C(1 + |w|^p)
\]
for \( t \in I \) a.e., \( w \in \mathbb{R}^s \), \( x \in A \),

3. \( w_k \) tends to \( w_0 \) in the norm topology of \( L^p(I, \mathbb{R}^s) \),

4. the systems \( (P_k) \) and the sets \( M_k \), \( k = 0, 1, 2, \ldots \), satisfy assumptions 1 and 2 of Theorem 7,

then conditions (a) and (b) of Theorem 7 hold.

Next, let us consider a mixed case when the cost functional is of the form

\[
J_k(x, u) = \int \left( \langle F^1(t, x(t)), w_k(t) \rangle + \langle F^2(t, v_k(t), u(t)) \rangle dt \right) + l_k(x(T))
\]

where \( F^1 : I \times \mathbb{R}^n \rightarrow \mathbb{R}^r \), \( F^2 : I \times \mathbb{R}^r \times \mathbb{R}^m \rightarrow \mathbb{R}^m \), \( k = 0, 1, 2, \ldots \).

Similarly as the previous theorem one can prove

\[ \text{Theorem 9 If} \]

1. the cost functional is of form (14),

2. the function \( F^1 \) is measurable with respect to \( t \) and continuous with respect to \( x \) (similarly \( F^2 \)),

3. for any bounded set \( A \subset \mathbb{R}^n \), there exist \( \alpha(.) \in L^1(I, \mathbb{R}^+) \) and \( C > 0 \), such that
\[
|F^1(t, x)| \leq \alpha(t)
\]
\[
|F^2(t, v, x)| \leq C(1 + |v|^p)
\]
for \( t \in I \) a.e., \( x \in A \) and \( v \in \mathbb{R}^s \),

4. \( w_k \) tends to \( w_0 \) in the weak topology of \( L^p(I, \mathbb{R}^m) \) when \( q \in [1, \infty) \), or weakly∗ when \( q = \infty \), \( v_k \) tends to \( v_0 \) in the norm topology of \( L^p(I, \mathbb{R}^m) \),

5. the control systems are given by (8), (10) and assumptions (A1)-(A3) hold,

then

the optimal values and the sets of optimal processes satisfy conditions (a) and (b) of Theorem 7.

4 \text{ Remarks on linear systems and an example} \]

In the theory of automatic control and engineering, linear systems with quadratic cost functionals are investigated very frequently. Such systems are of the form

\[
\dot{x} = A_k(t)x + B_k(t)u, \quad x(0) = p_k,
\]

where \( A_k(.) \in L^2(I, \mathbb{R}^{n \times n}) \), \( B_k(.) \in L^2(I, \mathbb{R}^{n \times m}) \), \( x(.) \in H^1(I, \mathbb{R}^n) \), \( u(.) \in U_k = \{ u \in L^2(I, \mathbb{R}^m) : u(t) \in M_k \} \), \( C_k(.) \in L^1(I, \mathbb{R}^{n \times n}) \), \( D_k(.) \in L^1(I, \mathbb{R}^{n \times m}) \), \( l_k(.) \) and \( M_k \) satisfy conditions (A6), \( k = 0, 1, 2, \ldots \).

If we assume that \( D_k(t) \) is positively defined for \( t \in I \) a.e., then system (15)-(16) possesses at least one optimal process \( (x^*_k, u^*_k) \in X_k^* \times U_k^* \). Assume that \( p_k \rightarrow p_0 \) in \( \mathbb{R}^s \), \( l_k(.) \rightarrow l_0(.) \) uniformly on bounded sets, \( A_k \rightarrow A_0 \) and \( B_k \rightarrow B_0 \) in \( L^2 \), \( C_k \rightarrow C_0 \) and \( D_k \rightarrow D_0 \) in \( L^1 \). In this case, it is easy to see that to system (15)-(16) Theorem 7 is applicable. Moreover, if the matrices \( A_k, B_k \) are constant, \( M_k \) are some convex polyhedrons which satisfy the normality condition (cf. [4]) and the functions \( l_k(.) \) are convex, then the optimal process \( (x^*_k, u^*_k) \) is uniquely defined for \( k = 0, 1, 2, \ldots \), and \( (x^*_k, u^*_k) \) tends to \( (x_0^*, u_0^*) \) weakly in \( L^1 \times L^2 \).

This implies that \( x^*_k(.) \) tends to \( x_0^*(.) \) uniformly on \( I = [0, T] \), \( x^*_k(.) \rightarrow x_0^*(.) \) weakly in \( L^2(I, \mathbb{R}^2) \), and \( u^*_k(.) \) tends to \( u_0^*(.) \) weakly in \( L^2(I, \mathbb{R}^m) \).

\[ \text{Example 10 Consider a two-dimensional system with scalar controls} \]

\[
\begin{align*}
\dot{x}^1 &= x^2 + \cos(\alpha_k(x^1 + x^2)) - \left(1 + \alpha_k |x|^2\right)^{\frac{1}{2}} \\
x^1(0) &= p_k
\end{align*}
\]

(17)
\[ J_k(x, u) = \int_0^1 \left[ -x^2(t) + u(t) + \beta_k(t)((x^1(t))^3 + (x^2(t))^4) \right] dt \]
\[ + \alpha_k(x^1(1) - 4)^2, \quad k = 1, 2, \ldots, \]

where \( \alpha_k, \beta_k \to 0 \) in \( L^2(I, \mathbb{R}) \), \( u(t) \in [-1, 1] \), \( I = [0, 1] \). The above (quite theoretical) example is nonlinear if \( \alpha_k, \beta_k \neq 0 \) and it is rather impossible to find effectively an optimal process \((x^*_k, u^*_k)\) for this system. But this example satisfies all the assertions of Theorem 8. In the limit situation, we have

\[ \begin{aligned}
\dot{x}^1 &= x^2 \\
\dot{x}^1(0) &= 0 \\
\dot{x}^2 &= u \\
\dot{x}^2(0) &= 0
\end{aligned} \tag{19} \]

By applying the Pontriagin maximum principle (cf. [2], [5], [6]), it is easy to show that \( u^*_0(t) = -1, x^*_0(t) = (-\frac{1}{2}t^2, -t) \) is the unique optimal process for system (19)-(20). Let \( \varepsilon > 0 \) be an arbitrary number. Theorem 8 implies that any optimal trajectory \( x^*_k(\cdot) \) with sufficiently large \( k \) is close to the trajectory \( x^*_0(\cdot) = (-\frac{1}{2}t^2, -t) \), i.e. \( \max |x^*_k(t) - x^*_0(t)| < \varepsilon \), and the sequence of optimal controls \( u^*_k(\cdot) \) tends weakly to \( u^*_0(\cdot) \) in \( L^2(I, \mathbb{R}) \). Moreover, the sequence of derivatives \( \dot{x}^*_k(\cdot) \) tends to the function \((-t, -1)\) weakly in \( L^2(I, \mathbb{R}^2) \).

5 Formulation of the two-dimensional problem and basic assumptions

Further we shall present some sufficient conditions for stability of the two-dimensional continuous optimal control systems.

Let us consider the systems of hyperbolic equations

\[ \begin{aligned}
\frac{\partial x}{\partial y} (x, y) + A_1 \frac{\partial v}{\partial y} (x, y) + A_2 \frac{\partial \psi}{\partial y} (x, y) + A^k v(x, y) &= B^k u(x, y) \\
v(x, 0) &= \phi^k(x), \quad v(0, y) = \psi^k(y) \\
\text{for all } (x, y) &\in P^2 = [0, 1] \times [0, 1] \tag{21}
\end{aligned} \]

with the cost functionals

\[ J^k (v, u) = \int_{P^2} F^k (x, y, v(x, y), u(x, y)) \, dxdy \tag{22} \]

where \( A_1, A_2, A^k \in \mathbb{R}^{N \times N}, B^k \in \mathbb{R}^{N \times m}, \phi^k, \psi^k \in H^1 ([0, 1], \mathbb{R}^N) \) and \( \phi^k(0) = \psi^k(0) = c^k \) for \( k = 0, 1, 2, \ldots \).

The system (21) is considered in the space \( AC^2 \) of the absolutely continuous functions \( z = z(x, y) \) defined on \( P^2 \) such that \( z_{x, y} (x, y) \in L^2 (P^2, \mathbb{R}^N) \). We assume that \( u \in U = \{ u \in L^2 = L^2 (P^2, \mathbb{R}^m) : u(x, y) \in M \text{ a.e.} \} \) and the set \( M \) is convex and compact subset of \( \mathbb{R}^m \).

It is easy to see, that using the following substitution

\[ z(x, y) = v(x, y) - \phi^k(x) - \psi^k(y) + c^k \]

we get equivalent problem

\[ \begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} (x, y) + A_1^k \frac{\partial \phi}{\partial y} (x, y) + A_2^k \frac{\partial \psi}{\partial y} (x, y) + A^k z(x, y) &= B^k u(x, y) + b^k (x, y) \\
z(x, 0) &= z(0, y) = 0 \quad \text{for all } (x, y) \in P^2.
\end{aligned} \tag{23} \]

\[ J^k (z, u) = \int_{P^2} F^k (x, y, z(x, y), u(x, y)) \, dxdy, \tag{24} \]

where

\[ b^k(x, y) = -A_1^k \frac{d}{dx} \phi^k(x) - A_2^k \frac{d}{dx} \psi^k(y) + A^k \phi^k(x) - A^k \psi^k(y) + x^k c^k. \]

The problem (23)-(24) will be considered in the space \( AC^2 \) consisting all functions \( z \in AC^2 \) such that \( z(x, 0) = 0 \) and \( z(0, y) = 0 \) for all \( x, y \in [0, 1] \).

For system (23) the following theorem holds (cf. [7])

**Theorem 11** For any \( u \in L^2 (P^2, \mathbb{R}) \) and any \( k = 0, 1, 2, \ldots \) problem \( (23) \) possesses a unique solution \( z^k \in AC^2_0 (P^2, \mathbb{R}^N) \) given by the formula

\[ z^k(x, y) = \int_{0}^{x} \int_{0}^{y} R^k(s, t, x, y) (B^k u(s, t) + b^k(s, t)) \, dsdt, \]

where the function \( R^k \) (called Riemann function) has the form

\[ R^k(s, t, x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(s-x)^i (t-y)^j}{i! j!} T_{i,j}^k, \]

and the sequence \( T_{i,j}^k \) is defined by the recurrence formulae

\[ T_{i,j}^k = T_{i,j-1}^k A_1^k + T_{i-1,j}^k A_2^k - T_{i-1,j-1}^k A^k, \quad T_{0,0}^k = I, \quad T_{i,j}^k = 0 \text{ for } i = -1 \text{ or } j = -1, \tag{25} \]

for \( k = 0, 1, 2, \ldots \).

We shall impose the following assumptions

**(B0)** \( \phi^k \rightarrow \phi^0, \psi^k \rightarrow \psi^0 \) in the norm topology of \( H^1 ([0, 1], \mathbb{R}^N) \) i.e. \( \int_{0}^{1} \left( \frac{d}{dx} (\phi^k(x) - \phi^0(x)) \right)^2 \, dx + \| \phi^k(0) - \phi^0(0) \|^2 \rightarrow 0 \), similarly for \( \psi^k \), and \( b^k \rightarrow b^0 \) in \( L^2 (P^2, \mathbb{R}^m) \).
Let us recall, that the weak upper limit of the sequences of the sets \( V^k \subset X \) is defined as the set of all cluster points (with respect to the weak topology) of sequences \( (v^k) \) where \( v^k \in V^k \) for \( k = 1, 2, 3, \ldots \). We denote this set as \( \text{wLimsup} V^k \).

We have

**Theorem 13** If

1. the problems \( (P^k) \) given by (23)-(24) satisfy conditions (B0)-(B5),

2. the sequence of cost functionals \( J^k(x,u) \) tends to \( J^0(x,u) \) uniformly on the \( A \times U \) for any bounded set \( A \subset AC^0_2 \),

then

(a) there exists a ball \( B(0,\rho) \subset AC^0_2 \) such that \( Z^k \subset B(0,\rho) \) for \( k = 0, 1, 2, \ldots \), i.e. there exists \( \rho > 0 \) such that, for any admissible trajectory \( z^k \) of system (21) \( \| z^k \|_{AC^0_2} \leq \rho \),

(b) the sequence of optimal value \( m^k \) tends to optimal value \( m^0 \),

(c) the weak upper limit of the optimal sets \( Z^k \times U^k \subset AC^0_2 \times L^2 \) is a nonempty set \( Z_* \times U_* \) and \( \text{wLimsup} (Z^k \times U^k) = Z_* \times U_* \subset Z^0_\times \times U^0_\times \).

If the set \( Z^k \times U^k \) is singleton i.e. \( Z^k \times U^k = \{ z^k_* , u^k_* \} \), for \( k = 0, 1, 2, \ldots \), then \( z^k_* \) tends to \( z^0_* \) weakly in \( AC^0_2 \) and \( u^k_* \) tends to \( u^0_* \) weakly in \( L^2 \). (Let us remind that the weak convergence of \( z^k_* \) to \( z^0_* \) in \( AC^0_2 \) implies uniform convergence of the functions \( z^k_* \) to \( z^0_* \) on \( P^2 \)).

### 6 Main results for two-dimensional systems

Based ourselves on Theorem 13 one can prove the following sufficient conditions for stability of two-dimensional optimal control system.

**Theorem 14** Suppose that problem (23)-(24) satisfies assumptions (B0) - (B5) and for any bounded set \( A \subset \mathbb{R}^n \), there exists a sequence of function \( \gamma^k \in L^1 \left( P^2, \mathbb{R}^+ \right) \) such that

\[
|F^k(x,y,z,u) - F^0(x,y,z,u)| \leq \gamma^k(x,y)
\]

for a.e. \( (x,y) \in P^2 \), \( (z,u) \in A \times M \) and \( k = 0, 1, 2, \ldots \). Moreover we assume that \( \gamma^k \to 0 \) in \( L^1 \left( P^2, \mathbb{R}^+ \right) \).
(a) the sequence of optimal values \( m^k \) tends to optimal value \( m^0 \), as \( k \to \infty \),

(b) \( \text{wLimsup} \left( Z^k \times U^k \right) \subset Z^0 \times U^0 \) and
\[ \text{wLimsup} \left( Z^k \times U^k \right) \neq \emptyset. \]

In the next theorem we shall assume that the integrands \( F^k \) depend on some functional parameter. Namely, let
\[ P^2 \times \mathbb{R}^r \times \mathbb{R}^N \ni (x,y,\omega,z) \mapsto G_1 (x, y, \omega, z) \to \mathbb{R} \]
and
\[ P^2 \times \mathbb{R}^r \times \mathbb{R}^N \ni (x,y,\omega,z) \mapsto G_1 (x, y, \omega, z) \to \mathbb{R}^m \]
In this case the following theorem holds

**Theorem 15** If

1. \[ F^k (x, y, z, u) = G_1 (x, y, \omega^k (x, y), z) + \left\langle G_2 (x, y, \omega^k (x, y), z), u \right\rangle, \]
where \( \omega^k (\cdot) \in L^p (P^2, \mathbb{R}^r) \), \( p \geq 1 \) and \( G_i \) is measurable with respect to \( (x, y) \) and continuous with respect to \( (\omega, z) \) for \( i = 1, 2 \),

2. \( \omega^k \to \omega^0 \) as \( k \to \infty \) in the norm topology of \( L^p (P^2, \mathbb{R}^r) \),

3. for any bounded set \( A \subset \mathbb{R}^N \) there exists \( C > 0 \) such that
\[ |G_1 (x, y, \omega, z)| \leq C (1 + |\omega|^p) \]
for \( (x, y) \in P^2 \) a.e., \( \omega \in \mathbb{R}^r \), \( z \in A \),

4. the problem \( \left( P^k \right) \) given by (23)-(24) satisfies assumptions (B0)-(B5),

then conditions (a) and (b) of Theorem 14 hold.

Next, let us consider a mixed case when the cost functional is of the form
\[
J^k (z, u) = \int_{P^2} \left( \left\langle G_1 (x, y, z (x, y)), \omega^k (x, y) \right\rangle + \left\langle G_2 (x, y, v^k (x, y), z (x, y)), u (x, y) \right\rangle \right) dx dy, \tag{28}
\]
where \( G_1 : P \times P \times \mathbb{R}^N \to \mathbb{R}^r \), \( G_2 : P \times P \times \mathbb{R}^r \times \mathbb{R}^N \to \mathbb{R}^m \),
\( k = 0, 1, 2, \ldots \)
We have

**Theorem 16** If

1. the cost functional is of the form (28),

2. the function \( G_1 \) is measurable with respect to \( (x, y) \) and continuous with respect to \( z \) (\( G_2 \) analogously),

3. for any bounded set \( A \subset \mathbb{R}^N \) there exists \( \alpha (\cdot) \in L^p (P^2, \mathbb{R}^r) \) and \( C > 0 \) such that
\[ |G_1 (x, y, z)| \leq \alpha (x, y), \]
\[ |G_2 (x, y, v, z)| \leq (1 + |v|^p) \]
for \( (x, y) \in P^2 \) a.e., \( z \in A \) and \( v \in \mathbb{R}^r \),

4. \( \omega^k \) tends to \( \omega^0 \) in the weak topology of \( L^q (P^2, \mathbb{R}^r) \) when \( q \in [1, \infty) \), or weakly*-\( \star \) when \( q = \infty \); \( v^k \) tends to \( v^0 \) in the norm topology of \( L^p (P^2, \mathbb{R}^r) \),

5. the control systems are given by (23)-(24) and assumptions (B0)-(B5) hold,

then the optimal values and the sets of optimal process satisfy conditions (a) and (b) of Theorem 14.

**Example 17** Consider two-dimensional continuous optimal control system with variable parameters of the form
\[
\left\{ \begin{array}{l}

z_{xy} (x, y) + A^1_1 z_x (x, y) + (1 + A^2_1) z_y (x, y) = (1 + B^1_k) u (x, y), \\
z (x, 0) = \phi^k (x), \quad z (0, y) = \psi^k (y), \\
u (x, y) \in [0, 1],
\end{array} \right. \tag{29}
\]

\[ J^k (z, u) = \int_{P^2} \left[ (x - 2) z + \omega^k_1 (x, y) \phi_1 (x, y, z (x, y)) + \omega^k_2 (x, y) \phi_2 (x, y, u (x, y)) + \frac{1}{4} (1 - x) u (x, y) + 4xy \right] dx dy \to \min, \tag{30}
\]

where \( \phi^k, \psi^k \) are absolutely continuous functions, \( \phi_1 \) is continuous and \( \phi_2 \) continuous and convex with respect to \( u \in [0, 1] \), \( \phi (\cdot) \in L^2 (P^2, [0, 1]) \), \( z \in AC^2 (P^2, \mathbb{R}) \), \( \omega^k_1 (\cdot), \omega^k_2 (\cdot) \in L^1 (P^2, [-1, 1]) \). By Theorem 12 system (29)-(30) possesses at least one optimal solution but in general it is not easy to find an optimal process for this system. Suppose that all parameters of system (29)-(30) are small i.e. we assume that \( A^1_1, A^2_1, B^1 \to 0 \) in \( \mathbb{R} \); \( \phi^k, \psi^k \to 0 \) in \( AC^2 ([0, 1], \mathbb{R}) \), \( \omega^k_1, \omega^k_2 \to 0 \) in \( L^1 (P^2, \mathbb{R}) \) as \( k \to \infty \).

In the limit case we obtain the following system
\[
\left\{ \begin{array}{l}

z_{xy} (x, y) + z_y (x, y) = u (x, y), \\
z (0, y) = z (0, y) = 0
\end{array} \right. \tag{31}
\]
\[ J^k (z, u) = \int_{P^2} \left[ (x - 2) z (x, y) + \frac{1}{4} (1 - x) u (x, y) + 4xy \right] dx dy. \tag{32}
\]
By Theorem 12 the above system possesses an optimal process and applying extremum principle (cf. [8]) we are able to find effectively optimal solution \((z^0, u^0)\) and optimal value \(m^0\). In fact, the Lagrange’s function for system (31)-(32) is of the form

\[
\mathcal{L}(z,u) = \int_{x_2} [(x-2)z(x,y) + \frac{1}{4}(1-x)u(x,y) + 4xy + v(x,y) (z_{xy}(x,y) + y - u(x,y))] \, dx \, dy,
\]

where \(v \in L^2(P^2, \mathbb{R})\).

The extremum principle implies that \(\mathcal{L}_z(z^0, u^0) h = 0\) for any \(h \in AC^2_0\) and

\[
\mathcal{L}(z^0, u^0) \leq \mathcal{L}(z^*, u^*) \tag{34}
\]

for any admissible control \(u^*\).

Taking account (33) and integrating by parts we get

\[
\mathcal{L}_z(z^*, u^*) h = \int_{x_2} [(x-2) h(x,y) + v(x,y) (z_{xy}(x,y) + h_y(x,y))] \, dx \, dy = \int_{x_2} \left[ \int_y \frac{1}{x} \int_x (x-2) \, dx \, dy + v(x,y) \right] \, dx \, dy + \int_x v(x,y) \, dx \, h_{xy}(x,y) \, dx \, dy = 0
\]

for any \(h \in AC^2_0\).

Thus

\[
v(x,y) + \int_x v(x,y) \, dx + (1-y) \left( \frac{3}{2} + 2x - \frac{1}{2} x^2 \right) = 0.
\]

The above equation is of the Volterra type therefore there exists a unique solution \(v_0\) of this one. By direct calculation it is easy to check that \(v_0(x,y) = (1-x)(1-y)\).

The minimum condition (34) takes the form

\[
\int_{x_2} (1-x) \left[ (y - \frac{3}{4}) u^0_0(x,y) \right] \, dx \, dy \leq 0
\]

for all \((x,y) \in [0,1]\).

This implies that

\[
u^0_0(x,y) = \begin{cases} 1 & \text{for } x \in [0,1], \ y \in [0,\frac{3}{4}] \\ 0 & \text{for } x \in [0,1], \ y \in [\frac{3}{4},1] \end{cases} \tag{35}
\]

and by (31) and (32)

\[
z^0_*(x,y) = \begin{cases} (1-e^{-x}) & \text{for } x \in [0,1], \ y \in [0,\frac{3}{4}] \\ (1-e^{-x}) & \text{for } x \in [0,1], \ y \in [\frac{3}{4},1] \end{cases} \tag{36}
\]

and

\[
m^0 = J^0(z^0_*, u^0) = \frac{55}{64}.
\]

Applying Theorem 15 to our example we can state that for any \(k = 1, 2, 3, \ldots\) there exists at least one optimal process \((z^0_k, u^0_k)\) for system (29)-(30) and the sequence \((u^k)_{k \in N}\) tends to \(u^0\) weakly in \(L^2\), \((z^k)_{k \in N}\) tends to \(z^0\) weakly in \(AC^2_0\), where \(u^0_0\) and \(z^0_0\) are defined by (35) and (36) respectively. Moreover the sequence \((m^k)\) of the optimal values for systems (29)-(30) tends to \(m^0 = \frac{55}{64}\) and the sequence of the optimal trajectories \((z^k)\) tends to \(z^0_k\) uniformly on \(P^2\).

In this way we can state that in general is difficult to find an optimal solution for (29)-(30) but the process \((z^0_k, u^0_k)\) given by (35)-(36) and optimal value \(m^0 = \frac{55}{64}\) is a good approximation for \((z^0_k, z^0_k)\) and \(m^0\) for \(k\) sufficiently large.

References