Controllability of Discrete Linear Repetitive Processes - A Volterra Operator Approach

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Abstract

Repetitive processes are a distinct class of 2D systems of both theoretical and applications interest. They arise, for example, in the modeling of industrial processes such as long-wall coal cutting and are the essential starting point for the study of classes of iterative learning control schemes. The development of a 'mature' systems theory for these processes has been the subject of considerable research effort over the past two decades which has resulted in very significant progress on stability theory and resulting tests. In this paper, we use a Volterra operator setting to produce significant new results on the controllability of the sub-class of so-called discrete linear repetitive processes which are of particular interest in a number of areas, eg the modeling and analysis of a wide class of iterative learning control schemes.

1 Introduction

The essential unique characteristic of a repetitive (termed multipass in the early literature) process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of sweeps, termed passes, of the processing tool. Assume that the pass length \( h \) (i.e. the duration of a pass of the processing tool), which is finite by definition, has a constant value for each pass. Then in a repetitive process the output vector, or pass profile, \( y_k(t), 0 \leq t \leq h \) (\( t \) being the independent spatial or temporal variable) produced on pass \( k \) acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile \( y_{k+1}(t), 0 \leq t \leq h, k \geq 0 \).

Industrial examples (see, for example, [1]) include long-wall coal cutting and metal rolling operations. Also problem areas exist where adopting a repetitive process perspective has clear advantages over alternatives. This is especially true for classes of iterative learning control schemes [2] and of iterative solution algorithms for classes of nonlinear dynamic optimal control problems based on the maximum principle [3].

The basic unique control problem for repetitive processes is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the \( k \)-direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single-input single-output) repetitive processes and, in particular, long-wall coal cutting were based on first converting the system into an infinite-length single-pass process [4]. This resulted in a scalar differential/algebraic system to which standard scalar inverse-Nyquist stability criteria were then applied. In general, however, it was soon established that this approach to analysis and controller design would, except in a few very restrictive special cases, lead to incorrect conclusions [5]. The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature and the effects of resetting the initial conditions before the start of each pass.

To remove these difficulties, a rigorous stability theory has been developed [5, 6]. This theory is based on an abstract model of the dynamics in a Banach space setting which includes all processes with linear dynamics and a constant pass length as special cases. In effect, this theory consists of the distinct concepts of asymptotic stability and stability along
the pass, and necessary and sufficient conditions for these properties are available. These are expressed in terms of conditions on the bounded linear operator in the abstract model which describes the contribution of the previous pass dynamics to the current one.

The application of this stability theory to various subclasses of interest has also been reported ([6]). These include so-called discrete linear repetitive processes which are the subject of this paper. This sub-class has close links with 2D linear systems, i.e. information propagation in two separate directions, described by the well known and extensively studied Roesser [7] and Fornasini Marchesini [8] state space models. A key difference, which prevents direct application of systems theory developed for these 2D linear systems state space models, is that in repetitive processes the duration of information propagation in one of the two separate directions - along the pass - is finite by definition.

Some significant research has already been reported [9, 10] on the definition and characterization of systems theoretic properties for discrete linear repetitive processes. As in the 2D Roesser/Fornasini Marchesini processes (and multidimensional (nD), n \(>2\) systems in general), the situation is more complex than in the standard, termed 1D here, case. For example, there is more than one distinct concept of controllability for these processes.

These properties are also representation dependent - so-called local controllability (essentially, the existence of an admissible (in a well defined sense) control sequence which will drive the process to a pre-specified vector at a pre-specified point in its state space) can be completely characterized in terms of matrix rank based tests defined in terms of equivalent 2D linear systems state space models of the underlying dynamics [9]. Such an approach cannot, however, be used to completely characterize so-called pass (profile) controllability (essentially, the existence of an admissible (in a well defined sense) control sequence which will drive the process to a pre-specified pass profile on a pre-specified number). Instead, a 1D linear systems state space model again yields matrix rank based tests.

In this paper, we consider another practically relevant definition of controllability known as strong pass controllability which, in effect, demands the existence of an admissible (in a well defined sense) control sequence which will drive the process to a pre-specified set of pass profiles for all pass numbers greater than or equal to a pre-specified value. The solution is via the use of Volterra operator theory and the relationship with the previously studied pass controllability is established.

## 2 Background

Let \(E\) be a finite dimensional normed space over the complex field \(\mathbb{C}\) with norm \(|\cdot|_E\), and \(\mathbb{Z}_+\) be the set of nonnegative integers. Denote by \(s(\mathbb{Z}_+, E)\) the linear space of all sequences on \(E\), i.e. the functions \(f : \mathbb{Z}_+ \to E\). Then the set \(s(\mathbb{Z}_+, E)\) is a locally convex Hausdorff topological space when equipped with the topology of uniform convergence on finite sets, i.e. the family of neighborhoods is defined as

\[
U_{M, \varepsilon} = \{ f : f \in s(\mathbb{Z}_+, E), \ |f(k)|_E < \varepsilon, \ k \in M \},
\]

where \(M\) is the set of all finite subsets from \(\mathbb{Z}_+, \) and \(\varepsilon\) ranges over the set \(\mathbb{R}_+\) of all positive real numbers.

Denote by \(b(\mathbb{Z}_+, E)\) the subspace of \(s(\mathbb{Z}_+, E)\) consisting of all bounded functions, i.e. the functions \(f : \mathbb{Z}_+ \to E\) such that \(\sup_{k \in \mathbb{Z}_+} |f(k)|_E < +\infty\). Then the space \(b(\mathbb{Z}_+, E)\) is dense in the space \(s(\mathbb{Z}_+, E)\) with respect to the topology of uniform convergence over finite sets. The space \(b(\mathbb{Z}_+, E)\) can be converted into a normed space with, for example, the norm defined as \(\|f\| = \sup_{k \in \mathbb{Z}_+} |f(k)|_E\). Moreover, it is a Banach space.

Now let \(V\) and \(W\) be finite dimensional normed spaces over complex field \(\mathbb{C}\), \(A_1 : E \to E, A_2 : \mathbb{Z}_+ \to E, B_0 : W \to E, B : V \to W, D_0 : W \to W, D : V \to W\) be linear operators; \([0, h]\) the set of integers \(\{ i : 0 \leq i \leq h\}\), where \(h\) is the given integer. Then we can introduce the dynamics of discrete linear repetitive processes as:

\[
x(k+1,t+1) = A_1 x(k+1,t) + B_0 y(k,t) + B u(k+1,t), \quad (1)
\]

\[
y(k+1,t) = A_2 x(k+1,t) + D_0 y(k,t) + D u(k+1,t) \quad (2)
\]

with respect to the unknown functions \(x : \mathbb{Z}_+ \times [0, h] \to E, y : \mathbb{Z}_+ \times [0, h] \to W\).

The function \(u : \mathbb{Z}_+ \times [0, h] \to V\) is the control input vector, the function \(x : \mathbb{Z}_+ \times [0, h] \to E\) represents the state vector and \(y : \mathbb{Z}_+ \times [0, h] \to W\) represents the so-called pass profile vector - the process output. Also the solution of this model is defined as follows.

**Definition 1.** For a given control function \(u\) we say that the couple \(\{x(k,t), y(k,t)\}\) of functions defined on \(\mathbb{Z}_+ \times [0, h]\) with values in \(E\) and \(W\) respectively is the solution of equations (1) and (2) if they satisfy these equations for all \((k,t) \in \mathbb{Z}_+ \times [0, h]\).

It is easy to verify that for any function \(\alpha \in s(\mathbb{Z}_+, E)\) and for any collection of elements \(d_0, d_1, \ldots, d_h\) from \(W\) there is a unique solution to equations (1) and (2) such that the conditions

\[
x(k,0) = \alpha(k), \quad k \in \mathbb{Z}_+, \quad y(0,t) = d_t, \quad t \in [0, h]\]

hold. These are termed the initial conditions here.

### 3 Volterra operator and its properties.

The main goal of the research programme on which this paper is based is the use of Volterra operator methods in the solution of systems theoretic problems for linear repetitive processes. In this section, we establish the properties of the Volterra operator needed to solve the controllability problem considered in this paper.
Consider the operator \( V : b(\mathbb{Z}_+, E) \to b(\mathbb{Z}_+, E) \) defined by
\[
(Vf)(s) = \sum_{i=0}^{s} A_i f(s-i), \quad s \in \mathbb{Z}_+ 
\] (4)
where \( A_i : E \to V, \ i \in \mathbb{Z}_+ \) are given linear operators. Operators of this form are termed discrete Volterra operators [11, 12].

Consider now fixed bases in \( E \) and \( V \). Then the linear operators \( A_i \) can be interpreted as matrices on the complex field \( \mathbb{C} \). Also associate with each function \( x \in b(\mathbb{Z}_+, E) \) the analytic function \( x(z) \) defined by the power series \( x(z) = \sum_{i=0}^{\infty} x_i z^i \), where it is assumed that the sequence \( x_i \) converges in the unit disk \( D = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Then it can easily be shown that this mapping is bijective.

Now associate with each Volterra operator \( V \) its representation \( V(z) \) in the ring of power series defined by
\[
V(z) = \sum_{i=0}^{\infty} A_i z^i, \quad z \in \mathbb{C} 
\] (5)
Suppose now that the matrices \( A_i \) are such that the power series (4) converges in some domain containing the unit disk \( D \).

Then it follows immediately that \( \sum_{i=0}^{s} |A_i| < \infty \) and hence for each function \( f \in b(\mathbb{Z}_+, E) \) the following inequality holds
\[
|\sum_{i=0}^{s} A_i f(s-i)|_V \leq \left( \sum_{i=0}^{\infty} |A_i| \right) \| f \| 
\] (6)
Hence under these assumptions \( V \) is a bounded linear operator.

Let \( V_1, V_2 : b(\mathbb{Z}_+, E) \to b(\mathbb{Z}_+, E) \) be Volterra operators. Then the composition \( V_1 \cdot V_2 : b(\mathbb{Z}_+, E) \to b(\mathbb{Z}_+, E) \) is also a Volterra operator and its representation in the ring of power series corresponds to the multiplication of images : \( V_1 \cdot V_2 \rightarrow V_1(V_2(z)) \). Also if \( \alpha \in b(\mathbb{Z}_+, E) \) then the image \( V\alpha \in b(\mathbb{Z}_+, E) \) corresponds to the analytic function \( V(z) \alpha(z) \).

The following easily proven result now characterizes the inverse operator of \( V \).

**Lemma 1.** If \( E = V \) and \( \det V(z) \neq 0 \) for \( z \in \mathbb{D} \), then (i) the Volterra operator \( V \) is invertible, and (ii) \( V^{-1} \) is a Volterra operator.

It can also be shown that the matrix \( V(z) \) can be transformed ( or factored) by applying appropriate elementary operations to obtain the following
\[
V(z) = \sigma_1(z)p(z)\sigma_2(z), 
\] (7)
where \( \sigma_1(z) \) and \( \sigma_2(z) \) are square matrices of appropriate dimension which are analytic in the unit disk \( \mathbb{D} \) and have nonzero determinants in this disk. The matrix \( p(z) \), which has the same dimensions as \( V(z) \), has elements which are all zero except, possibly, on the leading diagonal where entries which are monic polynomials with roots in \( \mathbb{D} \) can occur. Without loss of generality, it is assumed that such nonzero diagonal elements \( p_1(z), ..., p_{l}(z) \) of matrix \( p(z) \) occur in the first \( l \) rows of \( p(z) \). They also have the property that each nonzero polynomial \( p_i(z) \) divides \( p_{i+1}(z), 1 \leq i \leq l - 1 \). Then, the matrix \( p(z) \) can be written in the form
\[
p(z) = \begin{pmatrix}
p_1(z) & 0 & \cdots & 0 \\
0 & p_2(z) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p_l(z) \\
0 & 0 & \cdots & 0
\end{pmatrix} 
\] (8)
Note that the Volterra operators \( Q_1 \) and \( Q_2 \) generated by the matrices \( \sigma_1(z) \) and \( \sigma_2(z) \), respectively, are invertible. The following result, again from [11], also holds.

**Lemma 2.** The Volterra operator \( V : b(\mathbb{Z}_+, E) \to b(\mathbb{Z}_+, E) \) is bijective if, and only if, \( \text{rank} V(z) = n \) ( \( n = \dim E \) \( \forall z \in \mathbb{D} \)).

Let \( A_i : E \to V, \ B_i : W \to V, \ i \in \mathbb{Z}_+ \) be linear operators, such that the power series \( V(z) = \sum_{i \in \mathbb{Z}_+} A_i z^i \), \( B(z) = \sum_{i \in \mathbb{Z}_+} B_i z^i \) are convergent in the unit disk \( \mathbb{D} \). Consider also the following system of equations
\[
\sum_{i=0}^{s} A_i x(s-i) + B_i y = \beta, \quad \forall s \in \mathbb{Z}_+ 
\] (9)
with respect unknown \( x \in b(\mathbb{Z}_+, E), y \in W \) with given \( \beta \in b(\mathbb{Z}_+, V) \) and suppose that \( \beta(z) \) is analytic in \( \mathbb{D} \). Then (8) is equivalent to the following equation for representations in the ring of power series
\[
V(z)x(z) + B(z)y(z) = \beta(z), \quad z \in \mathbb{D}. 
\] (10)

Let \( V(z) \) be an analytic \((n_1 \times n_2)\)-matrix \((n_1 = \dim E, n_2 = \dim V)\) and consider a row \( a(z) = (a_1(z), a_2(z), ..., a_{n_2}(z)) \) whose entries \( a_i(z) \) are analytic functions at \( z = z_0 \). Then this row is said to be \( k \)-annihilating \((k \geq 1)\) at \( z = z_0 \) for matrix \( V(z) \) if the \( n_2 \)-vector function \((z - z_0)^{-k}a(z)V(z)\) is analytic at \( z = z_0 \).

Let \( H_{z_0}^{k}[V] \) denote the set of all \( k \)-annihilating rows for \( V(z) \). In which case it is easy to verify that if \( a_1(z) \in H_{z_0}^{k}[V], a_2(z) \in H_{z_0}^{k}[V] \) then the sum \( a_1(z) + a_2(z) \) also belongs to \( H_{z_0}^{k}[V] \). Moreover, if the function \( \lambda(z) \) is analytic at \( z = z_0 \) then \( \lambda(z)a_1(z) \in H_{z_0}^{k}[V] \). Hence the set \( H_{z_0}^{k}[V] \) is a module over the ring of all analytic functions at \( z = z_0 \).

Suppose that the matrix \( V(z) \) is factored in the form (6) and let \( p_i \) denote the multiplicity of the root \( z = z_0 \) of the polynomial \( p_i(z) \) in (7). Then it is obvious that \( p_1 \leq p_2 \leq ... \leq p_l \). Also it can be shown that the rows \( \sigma_{1i}(z)(z - z_0)^{k-m_1}, ..., \sigma_{li}(z)(z - z_0)^{k-m_i}, \sigma_{1i+1}(z), ..., \sigma_{li+1}(z) \) form a basis for module \( H_{z_0}^{k}[V] \), where \( \sigma_{ij}^{-1}(z) \) denotes \( j \)-th row of the invertible matrix \( \sigma_{ji}^{-1}(z) \).

Let \( B(z) \) be an \((n_1 \times n_3)\)-matrix which is analytic at \( z = z_0 \) \((n_3 = \dim W)\) and denote by \( L_{z_0}[V, B] \) the linear span of all \( n_2 \)-dimensional rows of the form \([a(z)B(z)]_{z=z_0}^{(i)}, i = 0, 1, ..., k-1 \), where \( a(z) \in H_{z_0}^{k}[V], \)
the index $k$ takes the values $1, 2, \ldots$, and $[f(z)]^{(i)}_{z=z_0} = \frac{d^i f(z_0)}{dz^i}$ denotes the $i$-th derivative at the point $z = z_0$. Then the algebraic structure of the linear space $L_{z_0}[V, B]$ can be characterized in the following way. In each module $H^k_n[V]$ take a basis $h^k_1(z), \ldots, h^k_n(z)$, where, see above, the linear span of the $n_k$ - dimensional rows $[h^k_1(z)B(z)]_{z=z_0}, \ldots, [h^k_n(z)B(z)]_{z=z_0}$, $k = 1, 2, \ldots$ coincides with $L_{z_0}[V, B]$. Also if $n_1 \geq n_2$ and $l = n_2$ in (7), then $L_0[V, B]$ coincides with the linear span of the rows $\sigma^{-1}(z)B(z))^{(k)}_{z=z_0}$, $i = 1, \ldots, n_2$; $k = 0, \ldots, \rho - 1$. (11)

Introduce now $R_V = \{ z \in \mathbb{C}, |z| < 1, \text{rank}V(z) < n_2 \}$. Then this set either coincides with $D$ (when $n_1 = \text{dim} E_1 < \text{dim} E_2 = n_2$ or $n_1 \geq n_2$ and $p_{n_2} = 0$ in (7)), or else it contains a finite number of elements $z_1, \ldots, z_r$, which are the roots of equation $p_{n_2}(z) = 0$. Also denote the multiplicity of a root $z_i$ of polynomial $p_j(z)$ from (7) by $h_{ij}$ ($h_{ij} = 0$ when $p_j(z_i) \neq 0$) and set $\rho(V) = \sum_{j=0}^{n_2} \sum_{i=0}^{r} h_{ij}$. Then it is known that the number $\rho(V)$ is independent of transformations which reduce the matrix $V(z)$ to the factorization (6). (This number is also referred to as the singularity power of $V(z)$ in the unit disk $D$.)

We can state the following theorem.

**Theorem 1.** Equation (8) has a solution in the class $b(\mathbb{Z}_+, E)$ for any $\beta \in b(\mathbb{Z}_+, V)$ if, and only if,
(a) $\text{rank}V(z) = n_2$ ($n_2 = \text{dim} V$) for all $|z| = 1$; and
(b) $\text{dim}(L_{z_1}[V, B] + \ldots + L_{z_r}[V, B]) = \rho(V)$, $z_i \in R_V$, $i = 1, 2, \ldots, r$.

4 Strong Pass Controllability

The problem treated in this paper can be stated as follows.

**Definition 2.** The discrete linear repetitive process (1)-3 is said to be strongly pass controllable in the class $b(\mathbb{Z}_+, V)$ if \( \exists \) a pair \((k^0, t^0) \in \mathbb{Z}_+ \times [0, h] \) such that for any functions $\alpha^{(1)}$, $\alpha^{(2)}$ from $b(\mathbb{Z}_+, E)$ and any collection $d^{(1)} = (d_0^{(1)}, \ldots, d_n^{(1)})$, $d^{(2)} = (d_0^{(2)}, \ldots, d_n^{(2)})$ of elements from $W$ $\exists$ a bounded control function

$$ u^0(k, t), (k, t) \in \{(k, t) : 1 \leq k \leq k^0 \} \cup \{(k, t) : t \in [0, t^0]\} \] $$

such that the corresponding solution \( x(k, t, \alpha^1, d^1, u^0), y(k, t, \alpha^1, d^1, u^0) \) of (1) and (2) is bounded and the conditions

\[
x(k, t^0, \alpha^1, d^1, u^0) = \alpha^{(2)}(k), \quad k \geq k^0,
\]

\[
y(k^0, t, \alpha^1, d^1, u^0) = d^2_{t}^{(2)}, \quad t = t^0, t^0 + 1, \ldots, h
\]

hold.

Introduce the following

\[
X(k, p) := \sum_{t=0}^{h} x(k, t)p^t
\]

\[
Y(k, p) := \sum_{t=0}^{\infty} y(k, t)p^t, \quad p \in \mathbb{C}
\]

\[
U(z, t) := \sum_{k=0}^{h} u(k + 1, t)z^k,
\]

\[
U(k, p) := \sum_{t=0}^{h} u(k, t)p^t, \quad z \in \mathbb{C}.
\]

Then the solution of the system (1)-(3) can be written as

\[
X(z, t^0) = A^0(z)X(z, 0) + \sum_{k=0}^{t^0 - 1} A^{t^0 - 1 - k}(z)\beta(z)d^{(1)}_{t^0} + \sum_{k=0}^h b(z)U(z, i)
\]

\[
Y(k, p) = Bk^0(p)Y(0, p) + \sum_{k=1}^{k^0} Bk^0 - k(p)(\gamma(p)\alpha^{(1)}(k)) + a(p)U(k, p)
\]

\[
= \sum_{k=0}^{k^0} Bk^0 - k(p)(\gamma(p)p^{h+1}x(k, h + 1))
\]

(12)

where $A(z)$, $\beta(z)$, and $B(p)$ are defined as follows

\[
A(z) = A_1 + zB_0(I - zD_0)^{-1}A_2,
\]

\[
\beta(z) = B_0 + zB_0(I - zD_0)^{-1}D_0,
\]

\[
B(p) = D_0 + pA_2(I - pA_1)^{-1}B_0,
\]

and also

\[
b(p) = B + pB_0(I - pD_0)^{-1}D
\]

\[
\gamma(p) = A_2(I - pA_1)^{-1}
\]

\[
a(p) = D + pA_2(I - pA_1)^{-1}B.
\]

It is now clear that strong pass controllability is equivalent to the existence of a bounded control function $u(k, t)$ which satisfies the following set of equations:

\[
x(k^0 + k, t^0, \alpha^{(1)}, d^{(1)}, u) = \alpha^{(2)}(k^0 + k), \quad k \in \mathbb{Z}_+
\]

\[
y(k^0, t^0 + t, \alpha^{(1)}, d^{(1)}, u) = d^{(2)}_{t^0 + t}, \quad t = 0, 1, \ldots, h - t^0
\]

for some \((k^0, t^0)\) and for any function $\alpha^{(2)} \in b(\mathbb{Z}_+, E)$, and any collection $d^{(2)}$ of elements from $W$. Also no loss of generality arises from setting:

\[
\alpha^{(1)}(k) = 0, \quad k \in \mathbb{Z}_+, \quad d^{(1)}_{t} = 0, \quad t \in [0, h].
\]

Over the set of the formal power series of the form (4), define the following "shift" operator $\Psi \left[ \sum_{i=0}^{\infty} B_i z^i \right] =$
Also let \( A(z) = \sum_{i=0}^{\infty} A_i z^i \) be another power series of the form (4) that is convergent in \( \mathcal{D} \). Then it is routine to show that for shift operator \( \Psi \)

\[
\Psi^k[V(z)G(z)] = V(z)\Psi^k[G(z)] + \sum_{i=1}^{k} \Psi^{i}[V(z)]A_{k-i}, \quad k > 0 \tag{13}
\]

i.e. ‘multiplication by parts’ is a valid operation.

Over the set of the formal power series define the following “cut-off” operator \( \Gamma^h \left[ \sum_{i=0}^{\infty} B_i z^i \right] = \sum_{i=0}^{h} B_i z^i \). Then it follows immediately from definition 2 that

\[
\Psi^0 \left[ \sum_{i=0}^{h-1} A^{0-i-1}(z)b(z)U(z, i) \right] = \Psi^0 \sum_{k=0}^{h} \alpha^{(2)}(k)z^k, \tag{14}
\]

\[
\Gamma^h \left[ \Psi^0 \sum_{j=0}^{k-1} B^{j-1}(p)a(p)U(j+1, p) \right] = \Psi^0 \left[ \sum_{j=0}^{h} d^{(2)}_{j}p^j \right]. \tag{15}
\]

Applying (12) now yields

\[
\sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \Psi^{i+1} \left[ A^{0-i-1}(z)b(z) \right] u(k^0 - j, i) + A^{0-i-1}(z)b(z)\Psi^{k^0} U(z, i) \right] = \sum_{k=0}^{h} \alpha^{(2)}(k)z^k;
\]

\[
\Gamma^h \left[ \sum_{j=0}^{t-1} \sum_{i=0}^{t-1} \Psi^{i+1} \left[ B^{j-1}(p)a(p) \right] u(j+1, t^0 - i - 1) \right] + B^{j-1}(p)a(p)\Psi^{k^0} U(j+1, p) \right] = \sum_{j=0}^{h} d^{(2)}_{j}p^j. \tag{16}
\]

Also let

\[
M_{t^0}(z) = \{ b(z), A(z)b(z), \ldots, A^{0-i-1}(z)b(z) \},
\]

\[
N_{t^0}(p) = \{ a(p), B(p)a(p), \ldots, B^{0-i-1}(p)a(p) \},
\]

\[
U^{(1)}(z) = \{ \Psi^{k^0}[U(z, t^0 - 1)], \ldots, \Psi^{k^0}[U(z, 0)] \},
\]

\[
U^{(2)}(p) = \{ \Psi^0[U(k^0, p)], \ldots, \Psi^0[U(1, p)] \},
\]

\[
\Delta^{(2)}_{ij}(z) = \Psi^{i+1}[A^{0-j-1}(z)b(z)],
\]

\[
\Delta^{(2)}_{ij}(p) = \Psi^{i+1}[B^{k^0-j-1}(p)a(p)],
\]

\[
i = 0, 1, \ldots, t^0 - 1, \quad j = 0, 1, \ldots, k^0 - 1.
\]

Then (14) can be rewritten in the form

\[
M_{t^0}(z)U^{(1)}(z) + \sum_{j=0}^{k^0-1} \sum_{i=0}^{t^0-1} \Delta^{(2)}_{ij} u(k^0 - j, t^0 - 1 - i) = \sum_{k=0}^{\infty} \alpha^{(2)}(k^0 + k)z^k,
\]

\[
\Gamma^h \left[ N_{t^0}(p)U^{(2)}(p) + \sum_{j=0}^{k^0-1} \sum_{i=0}^{t^0-1} \Delta^{(2)}_{ij} u(k^0 - j, l^0 - 1 - i) \right] = \sum_{j=0}^{\infty} d^{(2)}_{j}p^j. \tag{17}
\]

For each \((t^0, k^0) \in \mathbb{Z}_+^2\), define

\[
\Delta^{(2)}_{k^0,t^0}(z) = \left( \Delta^{(2)}_{00}(z), \Delta^{(2)}_{01}(z), \ldots, \Delta^{(2)}_{k^0-1,t^0-1}(z) \right), \quad z \in \mathbb{C},
\]

\[
\Delta^{(2)}_{k^0,t^0}(p) = \left( \Delta^{(2)}_{00}(p), \Delta^{(2)}_{01}(p), \ldots, \Delta^{(2)}_{k^0-1,t^0-1}(p) \right), \quad p \in \mathbb{C}.
\]

Also assume that \( \sigma(A_1) \subseteq \mathcal{D}, \sigma(D_0) \subseteq \mathcal{D}, \) which guarantees that the matrices \( M_{t^0}(z), N_{t^0}(p) \) and \( \Delta^{(2)}_{k^0,t^0}(z), \Delta^{(2)}_{k^0,t^0}(p), i = 1, 2 \) are analytic functions in \( \mathcal{D} \). Hence, these matrices can be expanded to convergent (in \( \mathcal{D} \)) power series of the form

\[
\sum_{i=0}^{\infty} A_i^{(s)} z^i, \quad \sum_{s=0}^{\infty} B_i^{(s)} p^i, \quad s = 1, 2,
\]

where \( A_i^{(s)}, B_i^{(s)}, i \in \mathbb{Z}_+ \) are some linear operators defined on \( E^{k^0} \) and \( V^{k^0} \), respectively.

Now set

\[
y = [u(k^0, t^0 - 1), u(k^0 - 1, t^0 - 1), \ldots, u(1, t^0 - 1), \ldots, u(k^0, 0), u(k^0 - 1, 0), \ldots, u(1, 0)] \in V^{k^0}.
\]

\[
f^{(1)}(s) = [u(k^0 + s, t^0 - 1), \ldots, u(k^0 + s, 0)] \in V^{t^0},
\]

\[
f^{(2)}(t) = [u(k^0, t^0 + t), \ldots, u(1, t^0 + t)] \in V^{k^0},
\]

\[
\beta^{(1)} = (\alpha^{(2)}(k^0), \alpha^{(2)}(k^0 + 1), \ldots) \in b(\mathbb{Z}_+, E),
\]

\[
\beta^{(2)} = (d^{(2)}_{0}, d^{(2)}_{1}, \ldots, d^{(2)}_{h}), t = 0, 1, \ldots, h - t^0.
\]

Then (16) yields

\[
\sum_{i=0}^{\infty} A^{(1)}_{i} f^{(1)}(s - i) + B^{(1)}_{s} y = \beta^{(1)}_{s}, \quad s \in \mathbb{Z}_+.
\]

\[
\sum_{i=0}^{\infty} A^{(2)}_{i} f^{(2)}(t - i) + B^{(2)}_{t} y = \beta^{(2)}_{t}, \quad t = 0, \ldots, h - t^0. \tag{18}
\]

At this stage, we have established that strong pass controllability of (1)-(3) is equivalent to solvability of (17) (with respect to unknown \( f^{(1)}(s), f^{(2)}(t) \), \( y \) for an arbitrary function \( \alpha^{(2)}(s), s \in \mathbb{Z}_+ \) of \( b(\mathbb{Z}_+, E) \)), and for any finite collection of elements \( d^{(2)} = (d^{(2)}_{0}, \ldots, d^{(2)}_{h}) \).
In accordance with the notation of Section 3, introduce

\[ R_M = \{ z \in C, \ |z| < 1, \ \text{rank}M_{n_1}(z) < n_1 \} \] \hspace{0.5cm} (19)  
\[ R_N = \{ z \in C, \ |z| < 1, \ \text{rank}N_{n_2}(z) < n_2 \} \] \hspace{0.5cm} (20)

These sets both have a finite number of elements. Hence let \( R_M := \{ z_1, \ldots, z_m \} \) and \( R_N := \{ \omega_1, \ldots, \omega_n \} \) and now we have the following result.

**Theorem 2.** Let \( h \geq n_2 \), where \( n_2 = \dim V \). If the spectra \( \sigma(A_1) \) and \( \sigma(D_0) \) lie in the interior of the unit disk \( D \), then the linear repetitive process (1) - (3) is strongly pass controllable if, and only if,

(a) \( \text{rank}M_{n_1}(z) = n_1, \ \text{rank}N_{n_2}(z) = n_2 \) for all \( |z| = 1 \);  
(b) \( \exists \) a couple \((k^0, t^0)\) such that

\[
\dim \{ \mathcal{L}_{z_i}[M_{n_1}, \Delta^{(1)}_{k^0, t^0}] + \cdots + \mathcal{L}_{z_m}[M_{n_1}, \Delta^{(1)}_{k^0, t^0}] \\ + \mathcal{L}_{\omega_1}[N_{n_2}, \Delta^{(2)}_{k^0, t^0}] + \cdots + \mathcal{L}_{\omega_n}[N_{n_2}, \Delta^{(2)}_{k^0, t^0}] \} = \rho(M_{n_1}) + \rho(N_{n_2})
\]

where \( z_i \in R_M, \omega_j \in R_N \) are given by (18) and (19), and \( \rho(M_{n_1}), \rho(N_{n_2}) \) denote the singularity power of the matrices \( M_{n_1} \) and \( N_{n_2} \) respectively.

**Proof.** First introduce

\[
A_s = \begin{pmatrix} A^{(1)}_s & 0 \\ 0 & A^{(2)}_s \end{pmatrix}, \quad B_s = \begin{pmatrix} B^{(1)}_s \\ B^{(2)}_s \end{pmatrix}, \quad \beta_s = \begin{pmatrix} \beta^{(1)}_s \\ \beta^{(2)}_s \end{pmatrix}, \quad f_s = \begin{pmatrix} f^{(1)}_s \\ f^{(2)}_s \end{pmatrix}, \quad \text{where} \quad s = 0, 1, \ldots, h - t^0,
\]

\[
A_s = \begin{pmatrix} A^{(1)}_s & 0 \\ 0 & A^{(2)}_s \end{pmatrix}, \quad B_s = \begin{pmatrix} B^{(1)}_s \\ B^{(2)}_s \end{pmatrix}, \quad \beta_s = \begin{pmatrix} \beta^{(1)}_s \\ \beta^{(2)}_s \end{pmatrix}, \quad f_s = \begin{pmatrix} f^{(1)}_s \\ f^{(2)}_s \end{pmatrix}, \quad \text{where} \quad s > h - t^0.
\]

This now follows immediately from (17) that strong pass controllability of (1)-(3) is equivalent to solvability of the following equation

\[
\sum_{i=0}^{s} A_i f(s - i) + B_s y = \beta_s, \quad s \in \mathbb{Z}_+.
\]

Applying Theorem 1 to (20) now yields that \( f \) and \( y \) exist which solve this equation for arbitrary \( \beta_s, s \in \mathbb{Z}_+ \) if, and only if,

(i) \( \text{rank} \begin{pmatrix} M_{n_0}(z) \\ 0 \\ N_{k^0}(z) \end{pmatrix} = \text{dim}(E + V) = n_1 + n_2 \) for all \( |z| = 1 \); and

(ii) \( \dim \{ \mathcal{L}_{z_1}[M_{n_1}, \Delta^{(1)}_{k^0, t^0}] + \cdots + \mathcal{L}_{z_m}[M_{n_1}, \Delta^{(1)}_{k^0, t^0}] \\ + \mathcal{L}_{\omega_1}[N_{n_2}, \Delta^{(2)}_{k^0, t^0}] + \cdots + \mathcal{L}_{\omega_n}[N_{n_2}, \Delta^{(2)}_{k^0, t^0}] \} = \rho(M_{n_1}) + \rho(N_{n_2})
\]

where \( \{ z_1, \ldots, z_l \} \) is the set of \( z \) defined by the following condition

\[
R_{MN} = \{ |z| < 1, \ \text{rank} \begin{pmatrix} M_{n_0}(z) \\ 0 \\ N_{k^0}(z) \end{pmatrix} < n_1 + n_2 \}.
\]

Using the Cayley-Hamilton theorem and the assumption \( h \geq n_2 \), it follows that in condition (i) it can be assumed that \( t^0 \geq n_1 - 1 \) and that \( k^0 \geq n_2 - 1 \). Also it follows immediately that \( \text{rank}M_{n_1}(z) = n_1, \ \text{rank}N_{n_2}(z) = n_2 \) for all \( |z| = 1 \), i.e. condition (a) holds.

Consider now condition (b). Then it is obvious that \( R_{MN} = R_M \cup R_N \), where \( R_M \) and \( R_N \) are given by (18) and (19) respectively, and it is straightforward to verify that the following hold

\[
\mathcal{L}_{z_1} \begin{pmatrix} M_{n_0} & 0 \\ 0 & N_{k^0} \end{pmatrix}, \ \begin{pmatrix} \Delta^{(1)}_{k^0, t^0} \\ \Delta^{(2)}_{k^0, t^0} \end{pmatrix} = \mathcal{L}_{z_1} \begin{pmatrix} M_{n_0} & 0 \\ 0 & N_{k^0} \end{pmatrix} + \mathcal{L}_{z_1} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix} + \mathcal{L}_{z_1} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix} + \mathcal{L}_{z_1} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix} + \mathcal{L}_{z_1} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix}
\]

Also if \( z_i \notin R_M \), then \( \mathcal{L}_{z_i} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix} = 0 \) and similarly, if \( z_i \notin R_N \), then \( \mathcal{L}_{z_i} \begin{pmatrix} M_{n_1} & 0 \\ 0 & N_{k^0} \end{pmatrix} = 0 \). From definition of \( \rho(V) \) we have

\[
\rho \begin{pmatrix} M_{n_0} & 0 \\ 0 & N_{k^0} \end{pmatrix} = \rho(M_{n_0}) + \rho(N_{k^0}) = \rho(M_{n_1}) + \rho(N_{n_2}).
\]

Finally, applying these results to (ii) yields condition (b) of the theorem.

5 **Pass controllability**

In certain cases (see [10] for a full discussion) the following definition of so-called pass controllability plays a significant role.

**Definition 3.** The linear discrete repetitive process (1)-(3) is said to be pass controllable if \( \exists \) a pass number \( k^0 \in \mathbb{Z}_+ \) such that for any function \( \alpha^1 \) and any collection \( d^{(1)} = (d^{(1)}_0, \ldots, d^{(1)}_h), d^{(2)} = (d^{(2)}_0, \ldots, d^{(2)}_h) \) of elements from \( W \) \( \exists \) a control function

\[
u^0(k, t), (k, t) \in \{ (k, t) : 1 \leq k \leq k^0, \ 0 \leq t \leq h \}
\]

such that the corresponding solution \( y(k, t, \alpha^1, d^{1}, u^0) \) of (1)-(3) satisfies the following conditions

\[
y(k^0, t, \alpha^1, d^{1}, u^0) = d^{(2)}_t, \quad t = 0, 1, \ldots, h
\]

**Note 1.** No loss of generality arises here from assuming that

\[
\alpha^{(1)}(k) = 0, \quad k \in \mathbb{Z}_+, \quad d^{(1)}_t = 0, \quad t \in [0, h].
\]

**Note 2.** In comparison to the strong pass controllability of the previous section, definition 3 only focuses on a single pass \( k = k^0 \) rather than on all passes \( k \geq k^0 \).

Using (11), it follows now that the process (1)-(3) is pass controllable if there exists a number \( k^0 \in \mathbb{Z}_+ \) such that for any \( d^{(2)} = (d^{(2)}_0, \ldots, d^{(2)}_h) \) of elements from \( W \)

\[
\Gamma^h \sum_{j=0}^{k^0 - 1} B^k \delta_{j} \alpha(p) \mu(j + 1, p) = \sum_{j=0}^{k^0 - 1} d^{(2)}_j p^j
\]
holds.

Obviously, the functions $B(p), a(p)$ etc here are analytic in some neighbourhood, say $\Sigma$, of the point $p = 0$. If the spectra $\sigma(A_1)$ and $\sigma(D_0)$ lie in the interior of the unit disk $\mathbb{D}$, then these functions are also analytic in the disk $\mathbb{D}$. Hence we may consider the class $s(\mathbb{Z}^+, \mathbb{V})$ of all sequences $u : \mathbb{Z}^+ \to \mathbb{V}$ here in contrast to Section 4, where the class $b(\mathbb{Z}^+, \mathbb{V})$ of bounded sequences was needed. In which case we can omit the restrictions on the spectrums of $A_1$ and $D_0$.

**Theorem 3.** The linear repetitive process (1)-(3) is pass controllable in the class $s(\mathbb{Z}^+, \mathbb{V})$ if, and only if, $\exists k^0 \in \mathbb{Z}^+$ and $p^0 \in \Sigma$ such that

$$\text{rank} N_{k^0}(p^0) = n_3, \quad (n_3 = \dim W)$$

where

$$N_{k^0}(p) = \{a(p), B(p)a(p), ..., B^{k^0-1}(p)a(p)\}$$

**Proof.** **Necessity.** Assume that the process (1)-(3) is pass controllable for some $k^0 \in \mathbb{Z}^+$ and consider the following equation

$$\Gamma^h \left[ \sum_{j=0}^{k^0-1} B^{k^0-j-1}(p)a(p)U(j+1, p) \right] = \tilde{d}(p)$$

with respect to the unknown sequence $U(0, p), ..., U(k^0 - 1, p)$. Then since (1)-(3) is pass controllable, this equation is solvable for arbitrary analytic function $\tilde{d}(p)$ in $\Sigma$. Equation (23) can also be rewritten in the matrix form

$$\Gamma^h \left[ N_{k^0}(p) \tilde{U}(p) \right] = \tilde{d}(p)$$

where $\tilde{U}(p) = (U(0, p), ..., U(k^0 - 1, p))$. Now applying elementary operations to the polynomial matrix $N_{k^0}(p)$ yields that this matrix can be factored as

$$N_{k^0}(p) = \sigma_1(p) n_{k^0}(p) \sigma_2(p), \quad p \in \Sigma$$

Here $\sigma_1(p), \sigma_2(p)$ are invertible matrices in $\Sigma$, and $n_{k^0}(p)$ is an $(n_3 \times k^0 n_3)$-matrix of the form

$$n_{k^0}(p) = \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{27}$$

where, in general, $l \leq n_3$.

Suppose now that $\text{rank} N_{k^0}(p) < n_3 \forall p \in \Sigma$. Then it follows immediately that $l < n_3$ in (26), i.e. the matrix $n_{k^0}(p)$ has a zero row. Since $\sigma_1(p), \sigma_2(p)$ are invertible matrices in $\Sigma$, it follows that solvability of the equation

$$\Gamma^h \left[ N_{k^0}(p) \tilde{U}(p) \right] = \tilde{d}(p)$$

for arbitrary analytic function $\tilde{d}(p)$ in $\Sigma$. Since $n_{k^0}(p)$ has a zero row, this equation has no solution and a contradiction occurs.

**Sufficiency.** Suppose $\exists k^0 \in \mathbb{Z}^+$ and $p^0 \in \Sigma$ such that $\text{rank} N_{k^0}(p^0) = n_2$. Then obviously $\text{rank} N_{k^0}(p^0) = n_3 \forall p \in \Omega$, where $\Omega \subset \Sigma$ is some neighborhood of the point $p = p^0$. Hence $\forall p \in \Omega$ the matrix $n_{k^0}(p)$ can be factored as

$$n_{k^0}(p) = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{29}$$

In which case (27) has an analytic solution $\tilde{U}(p)$ in $\Omega$. It now follows immediately that (22) also has a solution for any polynomial $\tilde{d}(p)$ on its right-hand side, i.e. (1)-(3) is pass controllable.

**Remark.** In the paper by Galkowski et al [10] the concept of pass (profile) controllability was considered in a more stronger sense, i.e. it was assumed that the pass number $k^0$ is fixed. The following corollary of this last theorem gives the link between these two definitions.

**Corollary.** If (1)-(3) is profile controllable then it is pass controllable.

**Proof.** Theorem 9 from [10] yields that $\text{rank}(D, D_0, D_1, ..., D_{k^0-1}) = n_3$ if the process (1)-(3) is profile controllable. Put $p = 0$ in the formulas for $B(p)$ and $a(p)$ given earlier. Then $a(0) = D, B(0) = D_0$. Hence $\text{rank} N_{n_3}(0) = \text{rank}(D, D_0, D_1, ..., D_{k^0-1}) = n_3$, i.e. the process is pass controllable in the sense of [10] for $k^0 = n_3$.

### 6 Conclusions and Further Work

This paper has developed significant new results on the controllability of the distinct sub-class of 2D linear systems known as discrete linear repetitive processes. In common with other classes of 2D linear systems (and indeed nD linear systems), the situation somewhat more complicated than in the 1D case, i.e. there is more than one distinct definition of this property. Here we have used a Volterra operator approach to define and characterize so-called strong pass controllability for a model of the process dynamics defined in terms of linear operators defined on finite dimensional normed linear spaces over the complex filed $\mathbb{C}$. The link between this definition and controllability and that in previous work [10] has also been established.

In general terms, it is clear that this Volterra operator approach holds much promise for the solution of currently open systems theoretic research problems for discrete linear repetitive processes (and other sub-classes of interest such as so-called differential linear repetitive processes (where the along the pass dynamics are defined in terms of a continuous variable)). On obvious area for investigation is the formulation, and characterization of, definitions of observability for these processes.
By analogy with the case of controllability, it is to be expected that more than one distinct definition of observability will be required. On such definition is given next.

**Definition 4:** Consider the discrete linear repetitive processes model of (1)-(3) and introduce the so-called auxiliary state and output vectors for this process as

\[
\omega_1(k, t) = C_1 x(k, t), \quad \omega_2(k, t) = C_2 y(k, t)
\]

where \(C_1 : E \to Y_1 \), \(C_2 : W \to Y_2 \) are given linear operators, and \(Y_1, Y_2 \) are finite dimensional normed linear spaces over \(\mathbb{C} \). Then a discrete linear repetitive described by (1)-(3) is said to be strongly pass observable if \( \exists \) a couple \((k^0, t^0) \in \mathbb{Z}_+^2 \) such that the conditions

\[
C_1 x(k, t) = 0, \quad C_2 y(k, t) = 0, \quad (k, t) \in H_{k^0, t^0}
\]

where

\[
H_{k^0, t^0} = \{(k, t) \in \mathbb{Z}_+ \times [0, h], \ 0 \leq k \leq k^0 \} \cup \{(k, t) \in \mathbb{Z}_+ \times [0, h], \ 0 \leq t \leq t^0 \}
\]

yields

\[
\alpha^{(1)}(k) = 0, \quad k \in \mathbb{Z}_+, \quad d^{(3)}_t = 0, \quad t = 0, 1, ..., h
\]

It is to be expected that the Volterra operator based approach used here for controllability will extend in a natural manner to produce a characterization of this (and other) definitions/concepts of observability for discrete linear repetitive processes. This and other uses for this approach are currently under investigation.

**References**


