Spectral Factorization by Symmetric Extraction for Distributed Parameter Systems

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Abstract

The spectral factorization problem of a scalar coercive spectral density is considered in the framework of the Callier-Desoer algebra of distributed parameter system transfer functions. Criteria for the infinite product representation of a coercive spectral density and for the convergence of infinite product representations of spectral factors, i.e. for the spectral factorization problem, are described and commented. These theoretical results of convergence are illustrated by some examples.

1 Introduction

The spectral factorization problem plays a central role in feedback control system design, see e.g. [2], [13]; in particular, it is an essential step in the solution of the Linear-Quadratic optimal control problem for infinite-dimensional state-space systems, see e.g. [3], [4], [9], [12], [14] and the references therein.

This paper handles the description and convergence analysis of infinite product representations of spectral factors of a scalar coercive spectral density, in the framework of the Callier-Desoer algebra of distributed parameter system transfer functions (see e.g. [5],[8]). Criteria for the convergence of such infinite product representations are reported here, which extend some previous results (see [2],[4]). In particular, a criterion is described, which is based on the knowledge of the comparative asymptotic behavior of the spectral density poles and zeros; moreover a criterion for the infinite product representation of a coercive spectral density is proposed as a conjecture.

2 Preliminary Concepts and Results

Let \( \sigma \leq 0 \). An impulse response \( f \) is said to be in \( \mathcal{A}(\sigma) \) if for \( t < 0 \), \( f(t) = 0 \), and for \( t \geq 0 \), \( f(t) = f_a(t) + f_{sa}(t) \) where the regular functional part \( f_a \in \mathcal{L}_1(\sigma) \) and the singular atomic part \( f_{sa} = \sum_{i=0}^{\infty} f_i \delta(.-t_i) \), where \( t_0 = 0 \), \( t_i > 0 \) for \( i = 1,2,\ldots \) and \( f_i \in \mathcal{A} \) for \( i = 0,1,\ldots \) with \( \sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty \). The norm of a distribution \( f \) in \( \mathcal{A}(\sigma) \) is defined by

\[
\|f\|_{\mathcal{A}(\sigma)} := \int_0^\infty |f_a(t)| e^{-\sigma t} dt + \sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i}.
\]

An impulse response \( f \) is said to be in \( \mathcal{A}_- \) if \( f \in \mathcal{A}(\sigma) \) for some \( \sigma < 0 \). \( \mathcal{A}(\sigma) \) and \( \mathcal{A}_- \) are convolution algebras. \( \mathcal{A}_- \) (class of Laplace transforms of elements in \( \mathcal{A}_- \)) is our selected class of distributed proper-stable transfer functions. It contains the multiplicative subset \( \mathcal{A}(\infty) \) of transfer functions that are bounded away from zero at infinity in \( \mathcal{A}_+ \), i.e. that are biproper-stable.

Possibly unstable transfer functions are selected to be in the algebra \( \mathcal{B} \), where \( \hat{f} \in \mathcal{B} \) if \( \hat{f} = \hat{n} \hat{d}^{-1} \) with \( \hat{n} \in \mathcal{A}_- \) and \( \hat{d} \in \mathcal{A}(\infty) \). A transfer function is in \( \mathcal{B} \) iff it is the sum of a completely unstable strictly proper rational function and a stable transfer function in \( \mathcal{A}_- \); hence \( \hat{d} \) above can always be chosen biproper-stable rational. See e.g. [5], [8].

Definition 1 : A function \( \hat{F} \) is said to be a spectral density if \( \hat{F} \) is parahermitian self-adjoint, i.e. \( \hat{F} = \hat{F}_e = \hat{G}_e + \hat{G}^* \), where \( \hat{G} \) is in \( \mathcal{A}_- \) and \( \hat{G}_e(s) := \hat{G}(-\bar{s})^* \), and \( \hat{F} \) is non-negative on the imaginary axis, i.e. \( \hat{F}(j\omega) = \hat{F}_e(j\omega) = \hat{F}(j\omega)^* \geq 0 \) for all \( \omega \in \mathbb{R} \). A spectral density \( \hat{F} \) is said to be coercive if there exists \( \eta > 0 \) such that \( \hat{F}(j\omega) \geq \eta \hat{F} \) for all \( \omega \in \mathbb{R} \). A transfer function \( \hat{R} \) in \( \mathcal{A}_- \) is said to be a spectral factor of a spectral density \( \hat{F} \) if \( \hat{F}(j\omega) = \hat{R}_e(j\omega) \hat{R}(j\omega) \) for all \( \omega \in \mathbb{R} \). A spectral factor \( \hat{R} \) is said to be invertible if \( \hat{R}^{-1} \) is in \( \mathcal{A}_- \).

It is known that a spectral density has an invertible spectral factor iff it is coercive, see e.g. [2], [3], [6]. Moreover
spectral factors are unique up to multiplication by a constant of modulus one.

In the sequel all impulse responses \( f \) have no delayed impulses (delays), i.e. their singular atomic part is of the form \( f_{sa} = f_0 \delta ( \cdot ) \).

## 3 Infinite Product Representation of Spectral Densities

In the following conjecture, a criterion for the infinite product representation of coercive spectral densities is proposed.

**Conjecture 1**: [Criterion for the infinite product representation of a coercive spectral density] Let \( \hat{F} \) be a coercive spectral density given by \( \hat{F} = \hat{F}_s = G_s + \hat{G} \), where \( \hat{G} \in \mathcal{A} \) is such that \( G_{sa} = G_0 \delta ( \cdot ) \) for some \( G_0 \in \mathcal{A} \). Assume that the limit of \( \hat{F} \) at infinity exists in some vertical strip containing the imaginary axis such that

\[
\hat{F}(\infty) := \lim_{|\omega| \to \infty; -\delta < \sigma < \delta} \hat{F}(\sigma + j\omega) = 1.
\]

In addition, assume that \( \hat{F} \) is given by

\[
\hat{F}(s) = \frac{\hat{N}(s)}{\hat{D}(s)},
\]

where the denominator \( \hat{D} \) and numerator \( \hat{N} \) are entire functions such that \( \hat{D} = \hat{D}_s \) and \( \hat{N} = \hat{N}_s \), with real or complex conjugate zeros \( p_n, n = 1, 2, \ldots \) and \( z_n, n = 1, 2, \ldots \) respectively, where either \( z_n \) and \( p_n \in \mathbb{R} \), with \( z_n \) and \( p_n < 0 \), or \( z_n \) and \( p_n \in \mathbb{C} \), with \( \text{Im} \ z_n \) and \( \text{Im} \ p_n \neq 0 \), and \( \text{Re} \ z_n \) and \( \text{Re} \ p_n < 0 \), such that

\[
\sum_{n=1}^{\infty} \frac{1}{|p_n|^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty,
\]

and

\[
\sum_{n=1}^{\infty} \left| \left( \frac{p_n}{z_n} \right)^2 - 1 \right| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \left( \frac{z_n}{p_n} \right)^2 - 1 \right| < \infty.
\]

Moreover, for the case of complex conjugate zeros and poles, assume that

\[
\sum_{n=1}^{\infty} \left| \left( 2 + \frac{|z_n - p_n|}{|\text{Im} \ p_n|} \right) \cdot \frac{|z_n - p_n|}{|\text{Re} \ p_n|} \right| < \infty,
\]

and

\[
\sum_{n=1}^{\infty} \left| \left( 2 + \frac{|z_n - p_n|}{|\text{Im} \ z_n|} \right) \cdot \frac{|z_n - p_n|}{|\text{Re} \ z_n|} \right| < \infty.
\]

Then the spectral density \( \hat{F} \) has an infinite product representation of pole-zero pairs of the form

\[
\hat{F}(s) = \prod_{n=1}^{\infty} \hat{F}_n(s),
\]

and the inverse spectral density \( \hat{F}^{-1} \) has the infinite product representation of pole-zero pairs

\[
\hat{F}^{-1}(s) = \prod_{n=1}^{\infty} \hat{F}_n^{-1}(s),
\]

where the elementary factors \( \hat{F}_n \) are coercive rational spectral densities which are given either by

\[
\hat{F}_n(s) = \frac{z_n^2 - s^2}{p_n^2 - s^2}
\]

if \( z_n \) and \( p_n \in \mathbb{R} \), with \( z_n \) and \( p_n < 0 \), or by

\[
\hat{F}_n(s) = \frac{(z_n^2 - s^2)(\bar{z}_n^2 - s^2)}{(p_n^2 - s^2)(\bar{p}_n^2 - s^2)}
\]

if \( z_n \) and \( p_n \in \mathbb{C} \), with \( \text{Im} \ z_n \) and \( \text{Im} \ p_n \neq 0 \), and \( \text{Re} \ z_n \) and \( \text{Re} \ p_n < 0 \). □

**Comment 1**: Essential information towards the validity of this conjecture can be obtained from the Weierstrass factorization theorem for entire functions and Liouville’s theorem as in [4].

**Comment 2**: Conditions (2)-(5) are not the most general ones encountered in the theory of infinite product representation of entire functions: see e.g. [11, p. 358] and [15, Theorem 1, pp. 55-56]. However, according to [11], these conditions are applicable to many problems. Moreover these assumptions, together with the fact that a spectral density is parahermitian, lead to a simpler structure for the corresponding infinite product elementary factors. Conditions (3) mean that the spectral density poles and zeros should be asymptotically (i.e. as \( n \to \infty \)) sufficiently close.

## 4 Spectral Factorization by Symmetric Extraction

Now let us consider a coercive spectral density \( \hat{F} \) which is assumed to have an infinite product representation of pole-zero pairs of the form (6)-(9). Consider the standard invertible spectral factors \( \hat{R}_n \) of the spectral densities \( \hat{F}_n \), which are such that \( \hat{R}_n(\infty) = 1 \) and which are given by

\[
\hat{R}_n(s) = \frac{z_n - s}{p_n - s},
\]

(first order factor), when \( \hat{F}_n \) is defined by (8) and by

\[
\hat{R}_n(s) = \frac{(z_n - s)(\bar{z}_n - s)}{(p_n - s)(\bar{p}_n - s)},
\]

(second order factor), when \( \hat{F}_n \) is defined by (9), respectively.
A convergence criterion for the infinite product representation of spectral factors, i.e. for the symmetric extraction method of spectral factorization, is now reported which is based on the comparative asymptotic behavior of the spectral density poles and zeros, $p_n$ and $z_n$, as $n$ tends to infinity. This result can be seen as an extension of [4, Fact 1 and Theorem 5].

**Theorem 1:** [Spectral criterion for the convergence of the infinite product representation of a spectral factor] Consider a coercive spectral density $\hat{F}$ given by (6)–(9) for all $s$ in some vertical strip symmetric with respect to the imaginary axis. Let $\hat{R}_n$, $n = 1, 2, \ldots$, be the rational elementary factors defined by (10)–(11), with $2 \cdot |\sigma| \leq \min(|Re \ p_n|, |Re \ z_n|)$, for all $n$, for some $\sigma < 0$. Now, for the case that $\hat{R}_n$ is a first order factor given by (10), assume that there exists a constant $\alpha > 1$ such that

$$\frac{|z_n - p_n|}{|p_n|} = O\left(\frac{1}{n^\alpha}\right), \quad (12)$$

and

$$\frac{|z_n - p_n|}{|z_n|} = O\left(\frac{1}{n^\alpha}\right). \quad (13)$$

Moreover, for the case that $\hat{R}_n$ is a second order factor given by (11), assume that there exists a constant $\beta > 1$ such that

$$\frac{|z_n - p_n|}{|Re \ p_n|} = O\left(\frac{1}{n^\beta}\right), \quad (14)$$

and the sequence

$$\left(\sum_{n=1}^{N} \frac{|z_n - p_n|}{|Im \ p_n|}\right)_{N \geq 1}$$

is bounded, \quad (15)

and also

$$\frac{|z_n - p_n|}{|Re \ z_n|} = O\left(\frac{1}{n^\beta}\right) \quad (16)$$

and the sequence

$$\left(\sum_{n=1}^{N} \frac{|z_n - p_n|}{|Im \ z_n|}\right)_{N \geq 1}$$

is bounded. \quad (17)

Then the following assertions hold:

a) The infinite product in (6) converges to $\hat{F}$ in the Banach algebra $\mathcal{A}(\sigma) := \{ \hat{f} = \hat{f}_- + \hat{f}_+ : (f_-), \text{and } f_+ \in \mathcal{A}(\sigma) \}$ equipped with the norm

$$\|\hat{f}\|_\sigma := \|f\|_\sigma := \|(f_-)\|_{\mathcal{A}(\sigma)} + \|(f_+)\|_{\mathcal{A}(\sigma)}.$$

b) Any invertible spectral factor $\hat{R}$ in $\mathcal{A}_-$ of $\hat{F}$ is given by the infinite product representation

$$\hat{R}(s) = \xi \prod_{n=1}^{\infty} \hat{R}_n(s) = \xi \lim_{N \to \infty} \prod_{n=1}^{N} \hat{R}_n(s), \quad (18)$$

where $\xi$ is a complex number of modulus one and the limit is taken in the framework of the topology induced by the norm $\|\hat{f}\|_{\mathcal{A}(\sigma)} := \|f\|_{\mathcal{A}(\sigma)}$ on the Banach algebra $\mathcal{A}(\sigma)$. Hence, for any complex number $\xi$ of modulus one, the sequence

$$\langle \xi \cdot \prod_{n=1}^{N} \hat{R}_n \rangle_{N \geq 1} \quad (19)$$

of invertible approximate (rational) spectral factors converges to an exact invertible spectral factor $\hat{R} \in \mathcal{A}_-$ of $\hat{F}$ in the $\mathcal{A}(\sigma)$ norm, and the sequence

$$\langle \xi^{-1} \cdot \prod_{n=1}^{N} \hat{R}^{-1}_n \rangle_{N \geq 1} \quad (20)$$

converges to the corresponding inverse spectral factor $\hat{R}^{-1} \in \mathcal{A}_-$. \quad $\square$

5 **Concluding Remarks**

$\alpha$) The conclusions of Theorem 1 still hold if conditions (14)–(15) and (16)–(17) are replaced respectively by

$$(2 + \frac{|z_n - p_n|}{|Im \ p_n|}) \cdot \frac{|z_n - p_n|}{|Re \ p_n|} = O\left(\frac{1}{n^{3\beta}}\right), \quad (19)$$

and

$$(2 + \frac{|z_n - p_n|}{|Im \ z_n|}) \cdot \frac{|z_n - p_n|}{|Re \ z_n|} = O\left(\frac{1}{n^{3\beta}}\right). \quad (20)$$

$\beta$) Typically, in applications, e.g. LQ-optimal control or spectral factorization of a normalized coprime fraction spectral density (see e.g. [3], [4]) for an infinite–dimensional (without loss of generality) stable system, the $p_n$’s and $\bar{p}_n$’s are the poles of the open–loop transfer function and the $z_n$’s and $\bar{z}_n$’s are the poles of the closed–loop transfer function.

$\gamma$) The symmetric extraction method works very well for the heat diffusion equation, see [4], [1]. Indeed, in that case, the spectral density zeros and poles are all real, and the corresponding relative spectral errors $|z_n - p_n| \cdot |p_n|^{-1}$ and $|z_n - p_n| \cdot |z_n|^{-1}$ tend to zero exponentially fast as $n$ tends to infinity, whence (12) and (13) obviously hold for any $\alpha > 1$. The speed of convergence of the sequences (19) and (20) towards an invertible spectral factor $\hat{R}$ and its inverse $\hat{R}^{-1}$ respectively, is dictated by the magnitude of the parameters $\alpha$ and $\beta$ respectively. The larger these are, the better is the speed of convergence of the symmetric extraction method.

However this speed of convergence might not be as good as in the example mentioned above: see the limiting (academic) example in [7].

$\delta$) If there are a finite number of first order (second order) elementary factors $\hat{R}_n$, then only conditions (14)–(15) and (16)–(17) (respectively (12) and (13)) must be checked in Theorem 1.

$\epsilon$) The convergence of the spectral factor infinite product representation of Theorem 1 agrees with [10, Theorem 4.5]. By an independent analysis, it is possible to compute absolute and relative error estimates, in the $\mathcal{A}(\sigma)$ norm, for the spectral factor as well as for its inverse, especially when $\|\hat{R}_n\|_{\mathcal{A}(\sigma)}$ and $\|\hat{R}_n^{-1}\|_{\mathcal{A}(\sigma)}$ are of the order of the general term of a converging power series, see [4].
References


