Stabilization and nuclearity of fractional differential systems

Catherine Bonnet
INRIA Rocquencourt, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay cedex, France.
Tel: +33 1 39 63 51 40. Fax: +33 1 39 63 57 86
Catherine.Bonnet@inria.fr

Jonathan R. Partington
University of Leeds, School of Mathematics, Leeds LS2 9JT, U.K.
Tel.: +44 113 233 5123. Fax: +44 113 233 5145
J.R.Partington@leeds.ac.uk

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1 Introduction

We shall consider the class of systems with transfer functions of the form \( G(s) = h_2(s)/h_1(s) \), where \( h_2 \) and \( h_1 \) are in the ring of functions generated by fractional powers of \( s \) and the function \( \exp(-s^r) \) (for some fixed \( r \) with \( 0 < r < 1 \)). It is convenient to refer to such systems as fractional differential systems.

Many important infinite-dimensional systems have transfer functions that can be represented in the above form, that is, as functions \( G(s) \) which involve fractional powers of \( s \), possibly in combination with exponentials of fractional powers.

For instance, the following two PDE examples are given by Curtain [4], and some more elaborate examples can be found in [3] and [9].

(i) Heat equation with Neumann boundary control:
\[
G(s) = \frac{\cosh \sqrt{s}x_0}{\sqrt{s} \sinh \sqrt{s}};
\]
(ii) Heat equation with Dirichlet boundary control:
\[
G(s) = \frac{\sinh \sqrt{s}x_0}{\sqrt{s} \sinh \sqrt{s}}.
\]
In each case \( x_0 \) is a fixed number between 0 and 1.

Our third example, arising in the theory of noninductive cables, is given in [16, p. 321].

(iii) The transfer function \( G(s) = \frac{\exp(-as\sqrt{s})}{s} \) (where \( a > 0 \) is a constant), associated with the response of an infinitely long cable to a step voltage.

Matignon [10] studies fractional systems with transfer function \( G(s) = Q(s)/P(s) \), where \( Q \) and \( P \) are 'generalized polynomials', namely linear combinations of a finite number of powers of the form \( s^{\alpha_j} \), where \( \alpha_j > 0 \) is not necessarily an integer. Another reference for fractional differential equations is [11]. Closely related are the systems with diffusive representation, as analysed by Montseny [12].

We analyse several fundamental questions for this class of systems. We give necessary and sufficient conditions for nuclearity of such systems (an important concept in model reduction [5]), and for their bounded-input bounded-output (BIBO) stability; this enables us to resolve affirmatively a conjecture of Matignon [10] by proving a rather more general result. Second, we construct coprime factorizations and associated Bézout factors for these systems: these allow one to obtain a formula for all stabilizing controllers of these systems, using the well-known Youla parametrization (see e.g. [3]).

We remark that similar results for the class of retarded delay systems are already known: results on nuclearity were first given in [14] (more precise estimates appear in [6, 7]), and explicit formulae for Bézout factors can be found in [1, 13]. For the functions we consider here, there is the added difficulty that there is often a branch point at \( s = 0 \).

Recall that \( \mathcal{A} \) denotes the space of distributions of the form
\[
h(t) = h_a(t) + \sum_{i=0}^{\infty} h_i \delta(t - t_i) \quad \text{where} \quad t_i \in [0, \infty), \quad 0 \leq t_0 < t_1 < \ldots, \quad \delta(t - t_i) \text{ is a delayed Dirac function,} \quad h_i \in \mathbb{C}, \quad h_a \in L^1 \text{ and } \sum_{i=0}^{\infty} |h_i| < \infty.
\]
The norm on \( \mathcal{A} \) is defined by
\[
\|h\|_{\mathcal{A}} = \|h_a\|_{L^1} + \sum_{i=0}^{\infty} |h_i|.
\]
In this paper, we shall in fact work in the subspace of \( \mathcal{A} \) of distributions in \( L^1 + \mathbb{C} \delta \). \( \mathcal{A} \) denotes the space of Laplace transforms of functions in \( \mathcal{A} \).

We recall that BIBO-stability of a system \( P \) with convolution kernel \( h \) is defined as \( \sup_{u \in L^\infty, u \neq 0} \frac{\|h \ast u\|_{L^\infty}}{\|u\|_{L^\infty}} < \infty \) which is equivalent to \( \|h\|_{\mathcal{A}} = \|P\|_{\mathcal{A}} < \infty \).

A system with transfer function \( P \) is said to have a coprime factorization \( (N, D) \) over \( \mathcal{A} \) if \( P = ND^{-1} \), \( D \neq 0 \), \( N \), \( D \in \mathcal{A} \) and there exists \( X, Y \in \mathcal{A} \) such that \(-NX + DY = 1\).
$P$ analytic in $\{\Re s > 0\}$ and continuous on $i\mathbb{R}$ is said to be proper on $\{\Re s \geq 0\}$ if for sufficiently large $\rho$, 
\[ \sup_{\{\Re s \geq 0, |s| \geq \rho\}} |P(s)| < \infty; \] to be strictly proper on $\{\Re s \geq 0\}$ if
\[ \lim_{\rho \to \infty} \left( \sup_{\{\Re s \geq 0, |s| \geq \rho\}} |P(s)| \right) = 0; \] to have a limit at infinity in $\{\Re s \geq 0\}$ if there exists a complex constant $P_\infty$
such that
\[ \lim_{\rho \to \infty} \left( \sup_{\{\Re s \geq 0, |s| \geq \rho\}} |P(s) - P_\infty| \right) = 0. \]

Let $P$ be a function that is meromorphic in $\mathbb{C} \setminus \mathbb{R}^-$ and has
a branch point at $s = 0$. The point $s = 0$ is defined to be a pole of fractional order $\alpha > 0$ of $P$ if \( \lim_{s \to 0} |P(s)| = \infty \) and $\alpha$ is the unique real number such that \( \lim_{s \to 0} |s^\alpha P(s)| \) is finite
and nonzero.

**Remark 1.1** The function $e^{-s^r}$ where $r$ is a real number, $0 < r < 1$, has a branch point at $s = 0$. To study this function we make a cut in the complex plane at $\mathbb{R}^-$ and consider the domain $\mathbb{C} \setminus \mathbb{R}^-$. In this domain, we can extend the function by continuity at $s = 0$, so that the function is analytic in $\{\Re s > 0\}$ and continuous on $i\mathbb{R}$. In fact the domain of analyticity is even bigger than $\{\Re s > 0\}$ and it will be useful to exploit the fact that the function can be defined to be analytic and proper in $\mathcal{D} = \{s \in \mathbb{C}, s \neq 0, \text{ such that } |\arg s| < a\pi\}$ with $a > \frac{1}{2}$ and $ar < \frac{1}{2}$.

## 2 Stability of fractional systems

We consider systems with transfer functions of the type

\[ P(s) = \frac{\sum_{i=1}^{i=n_2} a_i(s)e^{-\beta_i s^r}}{\sum_{i=1}^{i=n_1} p_i(s)e^{-\gamma_i s^r}} = \frac{h_2(s)}{h_1(s)} \]  

where $r$ is a real number such that $0 < r < 1$, the $p_i$ are of the form $\sum b_k s^{\delta_k}$ with $\delta_k \in \mathbb{R}^+$ and the $a_i$ are of the form $\sum a_k s^{\alpha_k}$ with $\alpha_k \in \mathbb{R}^+$. Let us study their behaviour at zero and infinity. As $s \to 0$ we can write

\[ h_1(s) = s^\alpha(c_1 + o(1)) \quad \text{and} \quad h_2(s) = s^\beta(c_2 + o(1)), \]

with $\alpha, \beta \geq 0$. As $s \to \infty$ in the right-hand half-plane, we can write

\[ h_1(s) = s^\gamma(c_3 + o(1)) \quad \text{and} \quad h_2(s) = s^\delta(c_4 + o(1)), \]

with $\gamma, \delta \geq 0$. As we consider strictly proper systems (deg $p_0 > \deg q_0$) we have that $\gamma > \delta$.

We begin by observing that, although $P$ is not in general meromorphic on $\mathbb{C}$, it is meromorphic in a neighbourhood of every point of the closed right half plane except for 0.

**Proposition 2.1** Let $a$ satisfy $a > 1/2$ and $ar < 1/2$. Then the function $P$ is meromorphic in $\mathcal{D} = \{s \in \mathbb{C}, s \neq 0, \text{ such that } |\arg s| < a\pi\}$ (indeed, $h_1$ has a finite number of zeroes in $\mathcal{D}$).

Our next result answers positively a question of Matignon [10], and, indeed, gives a more general result. Recall that a system is said to be of nuclear type, if its sequence $(\sigma_n)$ of Hankel singular values satisfies $\sum \sigma_n < \infty$. (See, for example, [5].)

**Theorem 2.2** Under the hypotheses above, the following are equivalent:

(i) $P$ is a system of nuclear type;

(ii) $P$ is BIBO stable;

(iii) $P$ has no poles in $\{\Re s \geq 0\}$ (which implies that $\beta \geq \alpha$).

This result can be proved using the Coifman–Rockberg characterization of nuclearity [15]. The equivalence of (ii) and (iii) could also be established by the Hardy–Littlewood inequality [8], as was done in [2]. Note that (i) always implies (ii) for strictly proper systems [5].

## 3 Coprime factorization and Bézout factors

In this section we show how to construct coprime factorizations for the fractional differential systems under consideration, together with the associated Bézout factors. All the functions we work with can be expressed in terms of fractional powers of $s$ and $\exp(-s^r)$, so that we know about their stability properties from Section 2.

**Proposition 3.1** A coprime factorization $(N, D)$ of $P$ in $\hat{\mathbb{A}}$ is given by

\[ N(s) = \frac{(s + 1)^i(s^\mu + 1)h_2(s)}{s^\mu(s + 1)^\gamma(s^\gamma + 1)}, \]

\[ D(s) = \frac{(s + 1)^i(s^\mu + 1)h_1(s)}{s^\mu(s + 1)^\gamma(s^\gamma + 1)}, \]

where $\mu = \min(\alpha, \beta)$, $\lfloor \mu \rfloor$ denotes the integer part of $\mu$ and $\{\mu\}$ the fractional part, so $\mu = \lfloor \mu \rfloor + \{\mu\}$.

Note that $N, D$ are stable functions with the same form as (1).

Our final result gives a construction of explicit Bézout factors $X$ and $Y$ associated with the coprime factorization.
that $f(0, \ldots, 0)$, and the coefficients then the set of all stabilizing controllers can be parametrized by $K = (X + DQ)/(Y + NQ)$, where $Q \in \mathcal{A}$, in accordance with the Youla parametrization (cf. [3]). We stress here that we are analysing the most general case: often some simplifications are possible in specific examples.

**Theorem 3.2** Let $\sigma_1, \ldots, \sigma_m$ be the $m$ nonzero unstable zeroes of $h_1$ and let

$$
T_1(s) = s^\nu(s + 1)^{\gamma_1}(s^{\gamma_1} + 1) \\
T_2(s) = (s + 1)^{\mu_1}(s^{\mu_1} + 1)h_2(s) \\
T_3(s) = (s + 1)^{\mu_1}(s^{\mu_1} + 1)h_1(s).
$$

Now, let us define

$$Y(s) = \frac{T_1(s) + T_2(s)X(s)}{T_3(s)} \quad \text{and} \quad X(s) = \frac{f_0 + f_{\lambda_1}s^{\lambda_1} + \cdots + f_{\lambda_n}s^{\lambda_n} + f_{M-m+1}s^{M-m+1} + \cdots + f_Ms^M}{(s + 1)^M}$$

where $\lambda_n \in \mathbb{R}$ and $M \in \mathbb{N}$ is chosen such that $M > \lambda_n + m$, the coefficients $f_0, f_{\lambda_1}, \ldots, f_{\lambda_n}$ are chosen in order to satisfy that $T_1(s) + T_2(s)X(s)$ is of fractional order $\alpha$ at zero, and the coefficients $f_{M-m+1}, \ldots, f_M$ are chosen so that $T_1(\sigma_i) + T_2(\sigma_i)X(\sigma_i) = 0$ for $1 \leq i \leq m$.

Then $(X, Y)$ are Bezout factors associated to the coprime factors $N$ and $D$ of $P$.

The idea is to construct $X \in \mathcal{A}$ such that $Y = (1 + NX)/D$ is also in $\mathcal{A}$. This is a somewhat complicated interpolation problem, because of the possible presence of a branch point at the origin.

**References**


