A Frequency-Domain Criterion for Absolute Stability of Infinite-Dimensional Systems with Applications to Low-Gain Control

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Abstract

We report on absolute stability results for well-posed infinite-dimensional systems which, in a sense, extend the well-known circle criterion to the case that the underlying linear system is the series interconnection of an exponentially stable, well-posed, infinite-dimensional system and an integrator and the nonlinearity $\phi$ satisfies a sector condition of the form $\langle \phi(u), \phi(u) - au \rangle \leq 0$ for some constant $a > 0$. These results can be used to show convergence and stability properties of low-gain integral feedback control applied to exponentially stable, well-posed, linear, infinite-dimensional systems subject to actuator nonlinearities. The class of actuator nonlinearities under consideration contains standard nonlinearities which are important in control engineering such as saturation and deadzone.

1 Introduction

Absolute stability problems and their relations to positive-real conditions have played a prominent role in finite-dimensional systems and control theory and have led to a number of important stability criteria for closed-loop systems obtained by applying unity feedback controls to linear dynamical systems subject to static input or output nonlinearities, see, for example, Aizerman and Gantmacher [1], Khalil [7], Lefschetz [8], Leonov et al. [9] and Vidyasagar [21]. Although there is some literature on absolute stability problems in infinite dimensions (for example, Bucci [3], Corduneanu [5], Leonov et al. [9], Logemann [10], Wexler [26], [27]), the number of results available in the literature is fairly limited, in particular for systems with unbounded control and observation.

In this paper, we report on recent results for a certain absolute stability problem for the class of well-posed infinite-dimensional systems. This class, which allows for considerable unboundedness in the control and observation, is well documented in the literature, see for example Curtain and Weiss [6], Salamon [17], [18], Staffans [19], [20] and Weiss [22] – [25]. The class of well-posed, linear, infinite-dimensional systems is rather general in the sense that it includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications.

Consider the feedback system shown in Fig. 1, where $G$ is the transfer function of an exponentially stable, well-posed, linear, infinite-dimensional system $\Sigma$ and $\phi: U \to U$ is a locally Lipschitz nonlinearity which, for some $a \geq 0$, satisfies the sector condition

$$\langle \phi(u), \phi(u) - au \rangle \leq 0, \quad \forall u \in U , \quad (1)$$

where $U$ denotes the input space of $\Sigma$ which is assumed to be a real Hilbert space.

Given $b > 0$, we study the absolute stability problem of finding conditions on $G$ such that the feedback system in Fig. 1 is stable for all locally Lipschitz $\phi$ satisfying (1) for some $a \in [0, b)$. In Section 2 we see that if $G(0)$ is invertible and the positive real condition

$$I + \frac{b}{2} \left( \frac{1}{s} G(s) + \frac{1}{s} G^*(s) \right) \geq 0, \quad s \in \mathbb{C} \text{ with } \Re s > 0$$

holds, then, for all locally Lipschitz $\phi$ satisfying (1) for some $a \in [0, b)$, the equilibrium of the closed-loop system shown in Fig. 1 is stable in the large. Moreover, under suitable extra assumptions on $\phi$, the equilibrium is semi-globally exponentially stable. These results extend, in a certain sense, a part of the well-known circle criterion, see Remark 2.2, part (e).
In Section 3 we indicate how the absolute stability results obtained in Section 2 can be applied to to the low-gain integral control problem illustrated in Fig. 2, where \( r \in U \) is the reference vector and \( k > 0 \) is the integral gain.

An example of a diffusion process with output delay illustrating our results is given in Section 4.

This paper does not contain any proofs. These can be found in [11].

Notation. For \( \alpha \in \mathbb{R}, \mathbb{C}_\alpha := \{ s \in \mathbb{C} \mid \text{Re} \, s > \alpha \} \); \( B(X_1, X_2) \) denotes the space of bounded linear operators from a Hilbert space \( X_1 \) to a Hilbert space \( X_2 \); we write \( B(X) \) for \( B(X, X) \); the Laplace transform is denoted by \( \mathcal{L} \).

2 Absolute stability results

In the following let \( \Sigma \) be a well-posed linear system with state space \( X \), input space \( U \), output space \( Y = U \) (all real Hilbert spaces), transfer function \( G \) and generating operators \( (A, B, C) \).

The operator \( A \) generates a strongly continuous semigroup \( T_t \) on the state-space \( X \), \( B : U \rightarrow X \) is an admissible input operator for \( T_t \) and \( C : X \rightarrow U \) is an admissible output operator for \( T_t \). Here \( X_{−1} \) denotes the completion of \( X \) with respect to the norm \( ||x||_{−1} := ||(sI − A)^{−1}x|| \), where \( s \in \varrho(A) \) (the resolvent set of \( A \)) and \( || \cdot || \) is the norm in \( X \) (different choices for \( s \) lead to equivalent norms). Moreover, \( X_1 := \text{dom}(A) \), endowed with the graph norm of \( A \). The unbounded operator \( A : \text{dom}(A) \subset X \rightarrow X \) can be extended to a bounded operator \( A : X \rightarrow X_1 \). This extension will also be denoted by \( A \). The generating operators determine the input-output behaviour of \( \Sigma \) up to an additive constant; more precisely, we have for all \( s, z \in \varrho(A) \)

\[
\frac{1}{s − z}(G(s) − G(z)) = −C(sI − A)^{−1}(zI − A)^{−1}B.
\]

If \( B \) and \( C \) are bounded operators, then the familiar formula

\[
G(s) = C(sI − A)^{−1}B + D
\]

holds, where \( D \) denotes the feedthrough operator of \( \Sigma \).

In state-space terms, the nonlinear system in Fig. 1 is described by

\[
\begin{align*}
\dot{x} &= Ax + B\phi(u), \quad x(0) = x^0 \in X, \quad (2a) \\
y &= C_L(x − (zI − A)^{−1}B\phi(u)) + G(z)\phi(u), \quad (2b) \\
\dot{u} &= −y, \quad u(0) = u^0 \in U, \quad (2c)
\end{align*}
\]

where \( z \in \varrho(A) \) and \( C_L \) denotes the Lebesgue extension of \( C \). In the case of bounded \( B \) and \( C \), the output equation (2b) simplifies to

\[
y = Cx + D\phi(u).
\]

We assume that \( \phi \) is locally Lipschitz, i.e., for every bounded set \( W \subset U \) there exists a constant \( l \geq 0 \) such that

\[
||\phi(u) − \phi(v)|| \leq l||u − v||, \quad \forall v, u \in W.
\]

For \( T \in (0, \infty) \), a continuous function

\[
[0, T) \rightarrow X \times U, \quad t \mapsto (x(t), u(t))
\]

is a solution of (2) if \( (x(·), u(·)) \) is absolutely continuous as a \( (X_{−1} \times U) \)-valued function, \( x(t) = (zI − A)^{−1}B\phi(u(t)) \in \text{dom}(C_L) \) for almost every \( t \in [0, T) \), \( (x(0), u(0)) = (x^0, u^0) \) and the differential equations in (2) are satisfied almost everywhere on \([0, T)\), where the derivative in (2a) should be interpreted in the space \( X_{−1} \). It is shown in [11] that for each \( (x^0, u^0) \in X \times U \), there exists a unique solution \( (x(·), u(·)) \) of (2) defined on a maximal interval \([0, T)\).

If \( T < \infty \), then

\[
\limsup_{t→T} (||x(t)|| + ||u(t)||) = ∞.
\]

In the case of bounded \( B \) and \( C \), existence on a maximal interval and uniqueness follow from well-known results on abstract Cauchy problems, see for example Pazy [16].

In the following we also assume that \( \phi \) is sector bounded, i.e., there exist numbers \( a \leq b \) such that

\[
<\phi(v) − av, \phi(v) − bv> \leq 0, \quad \forall v \in U. \quad (3)
\]

Let \( S[a, b] \) denote the set of all functions \( \phi : U \rightarrow U \) such that (3) holds. It is easy to show that (3) holds if and only if

\[
\left\|\phi(v) − \frac{a + b}{2}v\right\| \leq \frac{b − a}{2}\left\|v\right\|, \quad \forall v \in U.
\]

This shows in particular, that if \( \phi \in S[a, b] \), then the graph of \( \phi \) is contained in a (non-convex) cone with vertex at the origin. For \( a < b \) we define

\[
S[a, b] := \bigcup_{a ≤ c < b} S[a, c].
\]

Stability in the large

The zero solution of (2) is called stable in the large if: (i) for all \( (x^0, u^0) \in X \times U \), the solution of (2) exists on \( \mathbb{R}_+ \); and (ii) there exists a continuous, strictly increasing function \( p : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( p(0) = 0 \) and such that for any \( l > 0 \), the solution of \( (x(·), u(·)) \) of the initial value (2) satisfies

\[
||x^0|| + ||u^0|| \leq l \Rightarrow ||x(t)|| + ||u(t)|| \leq p(l), \quad \forall t \geq 0.
\]

The following theorem shows that a suitable positive real condition in terms of the transfer function \( G(s)/s \) will ensure that the zero solution of (2) is stable in the large if \( \Sigma \) is exponentially stable and \( \phi \in S[0, b] \).

Theorem 2.1 Suppose that \( \Sigma \) is exponentially stable, \( G(0) \) is invertible and \( \phi : U \rightarrow U \) is locally Lipschitz. Let \( (x^0, u^0) \in X \times U \). If there exists \( b > 0 \) such that

\[
I + 2\left(\frac{1}{s}G(s) + \frac{1}{s}G^*(s)\right) \geq 0, \quad \forall s \in \mathbb{C}_0, \quad (PR)
\]

and if \( \phi \in S[0, b] \), then the following statements hold:
(a) the zero solution of (2) is stable in the large;

(b) the solution \((x(\cdot), u(\cdot))\) of (2) satisfies
\[
\lim_{t \to \infty} \|x(t)\| = 0, \quad x \in L^2(\mathbb{R}_+, X),
\]
\[
\lim_{t \to \infty} \|\phi(u(t))\| = 0, \quad \phi \circ u \in L^2(\mathbb{R}_+, U);
\]

(c) under the extra assumption that \(\dim U = 1, u^\infty := \lim_{t \to \infty} u(t)\) exists, is finite and satisfies \(\phi(u^\infty) = 0\).

Remark 2.2 (a) Since \(\Sigma\) is exponentially stable, it follows that \(G(s)/s\) is holomorphic in \(\mathbb{C}_0\). Combining this with (PR) shows that the function \(I + \frac{b}{2}G(s)/s\) is positive real. As in the finite-dimensional case (see Anderson and Vongpanitlerd [2], pp. 53), it can be shown that (PR) holds if and only if \(G(0) = G^*(0) \geq 0\) and
\[
I + \frac{b}{2} \left( \frac{1}{i\omega} G(i\omega) - \frac{1}{i\omega} G^*(i\omega) \right) \geq 0, \quad \forall \omega \in \mathbb{R}\setminus\{0\}. \quad (4)
\]

In this context it is interesting to note that there exists \(b > 0\) such that (PR) holds if and only if \(G(0) = G^*(0) \geq 0\) (see Logemann and Townley [15]). Moreover, if \(A = A^*\), \(B\) is bounded, \(C = B^*\) and \(D = D^*\) (where \(D = G(0) + B^*A^{-1}B\) denotes the feedthrough operator), then (4) holds with \(b = \varepsilon^2/(M^2\|B\|^2)\), where \(\varepsilon > 0\) and \(M \geq 1\) are such that \(\|T_t\| \leq Me^{-\gamma t}\) for all \(t \geq 0\). If \(D = 0\), then a sufficient condition for \(G(0) = -B^*A^{-1}B\) to be positive semi-definite is that \(A\) has a complete family of eigenvectors (which, for example, is the case if \(A\) has compact resolvent).

(b) Note that assertion (a) implies the boundedness of the solution \((x(\cdot), u(\cdot))\) of (2) for all \((x^0, u^0) \in X \times U\).

(c) If \(\phi^{-1}(0) = \{0\}\) and \(\dim U = 1\), then it follows from a combination of assertions (a)-(c) that the zero solution of (2) is globally asymptotically stable.

(d) Some of the statements in Theorem 2.1 remain true for time-varying sector bounded nonlinearities. More precisely, let \(\phi : \mathbb{R}_+ \times U \to U, (t, v) \mapsto \phi(t, v)\) be continuous in \(t\) and locally Lipschitz in \(v\), uniformly in \(t\) on bounded intervals. Statement (a) of Theorem 2.1 remains true for all such \(\phi\) satisfying
\[
(\phi(t, v), \phi(t, v) - tv) \leq 0, \quad \forall (t, v) \in \mathbb{R}_+ \times U
\]
for some \(c \in [0, b]\); furthermore, apart from the convergence of \(\phi(t, u(t))\) to 0 as \(t \to \infty\), statement (b) remains true also.

(e) The relationship to the circle criterion: Consider the feedback system shown in Fig. 3, where \(H(s)\) is the (rational) transfer function of a \(m\)-input, \(m\)-output, exponentially stable, finite-dimensional system and \(\phi : \mathbb{R}^m \to \mathbb{R}^m\) is a static nonlinearity. The circle criterion says that if
\[
I + \frac{b}{2} (H(i\omega) + H^*(i\omega)) \geq 0, \quad \forall \omega \in \mathbb{R},
\]
then for all locally Lipschitz \(\phi \in \mathcal{S}(0, b)\), the closed-loop system shown in Fig. 3 is globally exponentially stable (see [7], p. 409 and [21], p. 227). Hence Theorem 2.1 can be considered as an extension of the circle criterion to the case where \(H(s)\) is of the form \(H(s) = G(s)/s\) with \(G\) being the transfer function of an exponentially stable well-posed infinite-dimensional system, i.e. the plant is the series interconnection of an exponentially stable well-posed infinite-dimensional system and an integrator.

![Figure 3](image-url)

(5) For exponentially stable single-input single-output systems with bounded control and bounded observation, the stability properties of the zero solution of (2) have been investigated in [26] under the assumption that \(\phi\) is locally Lipschitz, \(\phi_0(v) > 0\) for \(v \neq 0\) and \(\lim_{|v| \to \infty} \int_0^v \phi(w) \, dw = \infty\). It is shown in [26] that for all such \(\phi\), the zero solution of (2) is uniformly asymptotically stable in the large, provided that
\[
\Re G(i\omega) \geq \varepsilon > 0, \quad \forall \omega \in \mathbb{R}, \quad (5)
\]
i.e., \(G\) is positive-real in a strict sense. Trivially, there are many examples where (5) is not satisfied, whilst (PR) holds for some \(b > 0\).

In the following we introduce extra assumptions on the nonlinearity \(\phi\) which will guarantee that \(u(t)\) converges as \(t \to \infty\) in the case that \(\dim U > 1\).

Corollary 2.3 Suppose that the assumptions of Theorem 2.1 hold. If there exists \(b > 0\) such that (PR) is satisfied, if \(\phi \in \mathcal{S}(0, b)\) and if the extra assumptions
\[
(A1) \quad \phi^{-1}(\{0\}) \cap W \text{ is finite for any bounded set } W \subset U,
\]
\[
(A2) \quad \inf_{w \in W} \|\phi(w)\| > 0 \text{ for any bounded, closed and nonempty set } W \subset U \text{ with } \phi^{-1}(\{0\}) \cap W = \emptyset,
\]
are satisfied, then statements (a) and (b) of Theorem 2.1 hold. Moreover, the limit \(u^\infty := \lim_{t \to \infty} u(t)\) exists, is finite and \(\phi(u^\infty) = 0\).

Of course, assumption (A2) is automatically satisfied if \(\dim U < \infty\). Corollary 2.3 shows in particular that if \(\phi^{-1}(\{0\}) = \{0\}\) and (A2) holds, then the zero solution of (2) is globally asymptotically stable.

**Exponential stability**

Under the assumptions of Theorem 2.1 the zero solution of (2) is globally exponentially stable, provided that \(\phi \in \mathcal{S}(a, b)\) for some \(a \in (0, b)\) (i.e., the nonlinearity \(\phi\) is assumed to satisfy a more restrictive sector condition than in Theorem 2.1). In the following, let \(\omega(T)\) denote the exponential growth constant of the semigroup \(T_t\).
Theorem 2.4 Suppose that $\Sigma$ is exponentially stable, $G(0)$ is invertible and $\phi : U \to U$ is locally Lipschitz. If there exists $b > 0$ such that (PR) holds and if $\phi \in S(a, b)$ for some $a \in (0, b)$, then the zero solution of (2) is globally exponentially stable, that is, there exist $N \geq 1$ and $\nu \in (\omega(T), 0)$ such that for all $(x^0, u^0) \in X \times U$ the solution $(x(\cdot), u(\cdot))$ of (2) satisfies
\[
\|x(t)\| + \|u(t)\| \leq N e^{\nu t}(\|x^0\| + \|u^0\|), \; \forall t \geq 0.
\]

For $W \subset U$ and $a \leq b$, let $S_W[a, b]$ denote the set of all functions $\phi : U \to U$ such that
\[
\langle \phi(w) - aw, \phi(w) - bw \rangle \leq 0, \; \forall w \in W.
\]

For $a < b$ we define
\[
S_W[a, b] := \bigcup_{a \leq c < b} S_W[a, c].
\]

Of course, $S_W[a, b] = S[a, b]$ and $S_W[a, a] = S[a, b]$. Theorem 2.1 and Theorem 2.4 can be used to derive the following semi-global exponential stability result. The assumptions on the nonlinearity $\phi$ are more restrictive than in Theorem 2.1, but less restrictive than in Theorem 2.4.

Theorem 2.5 Suppose that $\Sigma$ is exponentially stable, $G(0)$ is invertible and $\phi : U \to U$ is locally Lipschitz and there exists $b > 0$ such that (PR) holds. If $\phi \in S[0, b]$ and $\phi$ satisfies the two extra assumptions
\[
(A3) \; \phi \in S_V[a, b] \text{ for some open set } V \subset U \text{ with } 0 \in V \text{ and some } a \in (0, b),
\]
\[
(A4) \; \inf_{w \in W} \|\phi(w)\| > 0 \text{ for any bounded, closed, nonempty set } W \subset U \text{ with } 0 \not\in W,
\]
then the zero solution of (2) is semi-globally exponentially stable, that is, for every $M > 0$, there exists $N \geq 1$ and $\nu \in (\omega(T), 0)$ such that for all $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \leq M$, the solution $(x(\cdot), u(\cdot))$ of (2) satisfies
\[
\|x(t)\| + \|u(t)\| \leq N e^{\nu t}(\|x^0\| + \|u^0\|), \; \forall t \geq 0.
\]

Of course, for finite-dimensional $U$, (A4) holds if $\phi^{-1}(\{0\}) = \{0\}$ (by the continuity of $\phi$).

3 Applications to low-gain integral control

In this section we apply the results in Section 2 to the so-called low-gain tracking problem described in the Introduction (see also Fig. 2). Let $\Sigma$ be as in Section 2 and assume that $U = Y = R$ (see [11] for a treatment of the multivariable case $U = Y = R^m$). Moreover, we assume that $\phi \in N(\lambda)$ for some $\lambda > 0$, where
\[
N(\lambda) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is nondecreasing and globally Lipschitz with Lipschitz constant } \lambda \}.
\]

Let $z \in \varrho(A)$ and consider the nonlinear system
\[
\begin{align*}
\dot{x} &= Ax + B\phi(u), \quad x(0) = x^0 \in X, \quad (6a) \\
y &= C_L(x - (z - A)^{-1}B\phi(u)) + G(z)\phi(u), \quad (6b) \\
\dot{u} &= k(r - y), \quad u(0) = u^0, \quad (6c)
\end{align*}
\]
where $r \in \mathbb{R}$ is the reference vector and $k \in \mathbb{R}$ is a gain parameter. We mention that in the case of bounded $B$ and $C$ the output equation (6b) simplifies to
\[
y = Cx + D\phi(u).
\]

The aim is to show that under suitable conditions on $\Sigma$, the error $e(t) = r - y(t)$, becomes small in some sense as $t \to \infty$.

If $G$ is holomorphic and bounded on $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$ (which for example is the case if $\Sigma$ is exponentially stable) and $G(0) > 0$, then it is not difficult to show that the following positive-real condition
\[
I + \frac{k}{2} \left( \frac{1}{s} G(s) + \frac{1}{s} G^*(s) \right) \geq 0, \; \forall s \in \mathbb{C}_0 \quad (7)
\]
holds for all sufficiently small $k > 0$, see Lemma 3.10 in [15]. We define
\[
K := \sup \{ k > 0 : (7) \text{ holds} \}.
\]
Let $\mathcal{M}$ denote the space of all finite signed Borel measures on $\mathbb{R}_+$.

Theorem 3.1 Assume that $\Sigma$ is exponentially stable and $G(0) > 0$. Let $\lambda > 0$, $\phi \in N(\lambda)$, $k \in (0, K/\lambda)$ and let $r \in \mathbb{R}$ be such that
\[
\phi^r := r/G(0) \in \im \phi.
\]

Then the solution $(x(\cdot), u(\cdot))$ of (6) is unique and exists on $\mathbb{R}_+$, and for each $u^r \in \phi^{-1}(\{\phi^r\})$, there exists $N > 0$ such that for all $(x^0, u^0) \in X \times \mathbb{R}$ and all $t \geq 0$
\[
\|x(t) - x^r\| + \|u(t) - u^r\| \leq N(\|x^0 - x^r\| + \|u^0 - u^r\|), \quad (8)
\]
where $x^r := -A^{-1}B\phi^r$. Moreover, the following statements hold:
\[
(a) \; \lim_{t \to \infty} \phi(u(t)) = \phi^r, \; \phi \circ u - \phi^r \in L^2(\mathbb{R}_+, \mathbb{R}); \\
(b) \; \lim_{t \to \infty} \|x(t) - x^r\| = 0, \; x - x^r \in L^2(\mathbb{R}_+, X); \\
(c) \; e := r - y \in L^2(\mathbb{R}_+, \mathbb{R}); \\
(d) \; the \; limit \; \lim_{t \to \infty} u(t) := u^\infty \; exists, \; is \; finite \; and \; satisfies \; \phi(u^\infty) = \phi^r; \\
(e) \; under \; the \; additional \; assumption \; that \; \Sigma^{-1}(G) \in \mathcal{M}, \; the \; error \; e \; satisfies \; e = e_1 + e_2, \; where \; e_1 \; is \; a \; bounded \; function \; with \; \lim_{t \to \infty} e_1(t) = 0 \; and \; e_2 \in L^2(\mathbb{R}_+, \mathbb{R}) \; for \; any \; \alpha > \omega(T)$; if additionally $T_{u^0}x^0 \in X_1$ for some $\theta^0 \geq 0$, then $\lim_{t \to \infty} e_2(t) = 0$. 

Remark 3.2 (a) Whilst statement (c) of Theorem 3.1 need not imply asymptotic tracking, it does imply that the error is small for large $t$ in the sense that for all $\delta, \varepsilon > 0$, there exists $\tau > 0$ such that
\[
\mu_L(\{t \geq \tau : |e(t)| \geq \delta\}) \leq \varepsilon,
\]
where $\mu_L$ denotes the Lebesgue measure on $\mathbb{R}_+$. 

(b) The assumption in statement (e) of Theorem 3.1 that $\mathcal{L}^{-1}(G) \in \mathcal{M}$ implies the regularity of $\Sigma$. However, this assumption is not very restrictive in the sense that it seems to be satisfied in all practical examples of exponentially stable well-posed systems. In particular, it is satisfied if $B$ or $C$ is bounded (see Lemma 2.3 in [12]). Statement (e) implies that $\lim_{t \to -\infty} e(t) = 0$, provided that $\mathcal{L}^{-1}(G) \in \mathcal{M}$ and $T_{\rho}x^0 \in X_1$ for some $t_0 \geq 0$.

(c) In applying Theorem 3.1 it is important to know the constant $K$ or at least a lower bound for $K$. It has been shown in [13] how $K$ can be obtained from frequency-response experiments performed on the linear part of the plant.

(d) If we replace the global Lipschitz assumption on $\phi$ by the weaker assumption that $\phi$ is locally Lipschitz and $\phi(\cdot) - \phi(0) \in \mathcal{S}[0, b]$ for some $b \geq 0$, then the conclusions of Theorem 3.1 remain true for all sufficiently small $k > 0$.

(e) We remark that Theorem 3.1 considerably improves the main result of [14] (see Theorem 3.3 in [14]) in the sense that Theorem 3.1 guarantees better asymptotic properties. In particular, the following parts of Theorem 3.1 are new: the stability property (8), the fact that $\phi \circ u - \phi^r \in L^2(\mathbb{R}_+, \mathbb{R})$ and $x - x^r \in L^2(\mathbb{R}_+, X)$ and statements (c) and (d). 

Under certain additional assumptions on $\phi$ and $r$, it can be shown that the variables $x$, $u$, and $y$ converge exponentially fast. This will be addressed in the next result, Theorem 3.3.

In order to state this theorem, we need some preparation.

Let $f \in \mathcal{N}(\lambda)$. The Clarke [4] generalized directional derivative $f^\circ(v; w)$ of $f$ at $v$ in direction $w$ is given by
\[
f^\circ(v; w) = \limsup_{\xi \to v} \frac{f(\xi + hw) - f(\xi)}{h}.
\]
Define $f^-(\cdot) := -f^0(\cdot; -1)$ (if $f$ is $C^1$ with derivative $f'$, then $f^- \equiv f'$). A point $v \in \mathbb{R}$ is said to be a critical point (and $f(v)$ is said to be a critical value) of $f$ if $f^-(v) = 0$.

Let $\mathcal{M}_\alpha$ denote the space of all signed Borel measures $\mu$ on $\mathbb{R}_+$ such that the exponentially weighted measure $E \mapsto \int_E e^{-\lambda t} \mu(dt)$ belongs to $\mathcal{M}$. Equivalently, $\mathcal{M}_\alpha$ is the space of all signed Borel measures $\mu$ on $\mathbb{R}_+$ such that $\int_0^\infty e^{-\alpha t} |\mu|(dt) < \infty$, where $|\mu|$ denotes the total variation of $\mu$.

Theorem 3.3 Assume that $\Sigma$ is exponentially stable and $G(0) > 0$. Let $\lambda > 0$, $\phi \in \mathcal{N}(\lambda)$ and $k \in (0, K/\lambda)$. Let $r \in \mathbb{R}$ be such that
\[
\phi^r := r/G(0) \in \mathcal{S}
\]
and $\phi^r$ is not a critical value of $\phi$. Let $u^r$ be the unique element in $\mathbb{R}$ such that $\phi(u^r) = \phi^r$ and set $x^r := -A^{-1}B\phi^r$.

Then, for given $M > 0$, there exist $N \geq 1$ and $\nu \in (\omega(T), 0)$ such that for all $(x^0, u^0) \in X \times \mathbb{R}$ with $|x^0 - x^r| + |u^0 - u^r| \leq M$ and for all $t \geq 0$, the solution $(x(t), u(t))$ of (6) satisfies
\[
\|x(t) - x^r\| + |u(t) - u^r| \leq N e^{\nu t}(\|x^0 - x^r\| + |u^0 - u^r|)
\]
and
\[
|\phi(u(t)) - \phi^r| \leq \lambda N e^{\nu t}(\|x^0 - x^r\| + |u^0 - u^r|).
\]
Moreover, for all $(x^0, u^0) \in X \times \mathbb{R}$ with $|x^0 - x^r| + |u^0 - u^r| \leq M$, the following statements hold:

(a) for any $\alpha > \nu$, $e := r - y \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$;

(b) under the additional assumption that $\mathcal{L}^{-1}(G) \in \mathcal{M}_\alpha$ for some $\alpha < 0$, the error $e$ satisfies $e = e_1 + e_2$, where $e_1 \in L^\infty_\beta(\mathbb{R}_+, \mathbb{R})$ for any $\beta \geq \max(\alpha, \nu)$, and $e_2 \in L^2_\beta(\mathbb{R}_+, \mathbb{R})$ for any $\beta > \omega(T)$; if additionally $T_{\rho}x^0 \in X_1$ for some $t_0 \geq 0$, then $e_2 \in L^2_\infty(\mathbb{R}_+, \mathbb{R})$ for any $\beta > \omega(T)$.

Remark 3.4 (a) Statement (b) shows that exponentially fast asymptotic tracking is guaranteed if $\mathcal{L}^{-1}(G) \in \mathcal{M}_\alpha$ for some $\alpha < 0$ and $T_{\rho}x^0 \in X_1$ for some $t_0 \geq 0$. Again, the assumption that $\mathcal{L}^{-1}(G) \in \mathcal{M}_\alpha$ for some $\alpha < 0$ is not very restrictive and seems to be satisfied in all practical examples of exponentially stable well-posed systems. In particular, this assumption is satisfied if $B$ or $C$ is bounded (see Lemma 2.3 in [12]).

(b) We mention that parts (c) and (d) of Remark 3.2 remain relevant in the context of Theorem 3.3.

(c) As compared to the main result in [14] (see Theorem 3.3 in [14]), Theorem 3.3 is entirely new: the issue of exponential decay is not addressed in [14].

4 Example: diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = (0, 1)$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity $\phi$ with Lipschitz constant $\lambda > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (at point $x_c \in I$, $x_c \geq x_b$). We formally write this diffusion process as
\[
z_x(t, x) = \kappa z_{xx}(t, x) + \delta(x - x_b)\phi(u(t)),
\]
\[
y(t) = z(t - h, x_c)
\]
for all $t > 0$.

For simplicity, we assume zero initial conditions:
\[
z(t, x) = 0, \quad \text{for all } (t, x) \in [-h, 0] \times [0, 1].
\]
This system was analyzed in the context of low-gain integral control in [13], [14].
With input \( \phi(u(\cdot)) \) and output \( y(\cdot) \), this example is a well-posed (in fact, regular) linear system with transfer function \( G \) given by
\[
G(s) = \frac{e^{-sh} \sinh \left( \frac{s}{b} \right) \sinh \left( (1-x_c) \frac{s}{\kappa} \right)}{\kappa \frac{s}{\kappa} \sinh \left( \frac{s}{\kappa} \right)}.
\]

From [13] we know that
\[
K = \sup \{ k > 0 \mid (7) \text{ holds} \} = \frac{1}{|G'(0)|} = \frac{6\kappa^2}{x_b(1-x_c)(6h\kappa + 1-x_b^2 - (1-x_c)^2)}.
\]

Note the dependence of \( K \) on the time-delay \( h \): the larger \( h \), the smaller \( K \). By Theorem 3.3, part (b), for each \( k \in (0, K/\lambda) \), the integral control
\[
u(t) = k \int_0^t [r - y(t)] \, dt
\]
guarantees exponentially fast asymptotic tracking of every constant reference value \( r \) such that
\[
\phi^r = \frac{r}{G(0)} = \frac{\kappa r}{x_b(1-x_c)} \in \text{im} \phi
\]
and \( \phi^r \) is not a critical value of \( \phi \). Note that \( \phi^r \) does not depend on \( h \). It is easy to see that if \( 0 \in \text{im} \phi \), then the range of values of \( r \) which can be tracked is maximal if \( x_b = x_c = 1/2 \).

For purposes of illustration, we adopt the following values
\[
\kappa = 0.1, \quad x_b = \frac{1}{3}, \quad x_c = \frac{2}{3}, \quad h = 1
\]
and we consider a nonlinearity \( \phi \) of saturation type, defined as follows
\[
u \mapsto \phi(u) := \begin{cases} 1, & u \geq 1 \\ u, & u \in (0, 1) \\ 0, & u \leq 0 \end{cases}.
\]

In this case, \( K = 243/620 \approx 0.3919 \) and \( \lambda = 1 \). The critical values of \( \phi \) are 0 and 1. For \( r = 1 \), we have
\[
\phi^r = \frac{r}{G(0)} = \frac{\kappa}{x_b(1-x_c)} = 0.9 \in [0, 1] = \text{im} \phi.
\]

In particular, \( \phi^r \) is not a critical value of \( \phi \). In each of the following three cases of controller gains
\[
(i) \ k = 0.39, \quad (ii) \ k = 0.26, \quad (iii) \ k = 0.13,
\]
Fig. 4 depicts the output behaviour of the system under integral control, while Figures 5 and 6 depict the corresponding control input and integrator state, respectively. Fig. 7 illustrates the evolution of the temperature profile \( z(t, \cdot) \) in case (i). These figures were generated using SIMULINK Simulation Software within MATLAB, wherein a truncated eigenfunction expansion, of order 20, was adopted to model the diffusion process.

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† Preprints in this series are available on the World Wide Web at the URL http://www.maths.bath.ac.uk/MATHEMATICS/preprints.html.