Single-Input Partial Pole-Assignment in Gyroscopic Quadratic Matrix and Operator Pencils

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Abstract

In this paper, we present a uniform solution to the partial pole-placement problem for gyroscopic matrix second-order and distributed parameter control systems. The distinctive features of the solution are that it is obtained by using the knowledge of only a few eigenvalues and the associated eigenvectors, and no transformation to a first-order system (in case of the matrix formulation) or finite-dimensional discretization (in case of distributed parameters formulation) is invoked.

1 Introduction

The vibrating structure such as beams, buildings, bridges, highways, large space structures, etc., are distributed parameter systems. While it is desirable to obtain a solution of a problem in its own natural setting of distributed parameter systems, very often, due to the lack of available computational techniques, in practice, such a system is discretized to a matrix second-order system using a finite elements or a finite differences technique [1, 3, 13, 14], and then the problem is solved for this discretized reduced-order model. A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

\[ M \ddot{x}(t) + (D + G) \dot{x}(t) + K x(t) = 0 \]  

where

- \( M = M^T \) is mass or inertia matrix
- \( D = D^T \) is damping matrix
- \( G = -G^T \) is skew-symmetric (gyroscopic) matrix
- \( K = K^T \) is stiffness matrix

The system represented by (1) is called damped gyroscopic system. The eigenvalues of the system (1) are the eigenvalues of the quadratic pencil: \( P(\lambda) = \lambda^2 M + \lambda (D + G) + K \).

To avoid resonance or combat undesirable effects of vibrations caused by a few “bad” eigenvalues of the system, one needs to reassign those few “bad” eigenvalues, leaving the rest unchanged, by using a suitable control force. This problem is known as the partial pole-placement problem. The partial pole assignment problem is a practical variation of the well-known pole assignment problem in control theory which is concerned with assigning a full spectrum to the closed-loop system. For a description of the latter problem and the state-of-the-art numerical methods for this problem see [6].

Let a control force of the form

\[ f = Bh(t) \]

where \( B \) is the input (control) matrix is applied to the vibrating structure and the control vector \( h(t) \) chosen as

\[ h(t) = F^T \dot{x}(t) + G^T x(t) \]

Then the closed-loop system corresponding to (1) is

\[ M \ddot{x} + (D + G - BF^T) \dot{x} + (K - BG^T)x = 0. \]

So, mathematically, the partial pole-placement problem is the problem of finding the matrices \( F \) and \( G \) such that a few “bad” eigenvalues of the closed-loop quadratic pencil

\[ P_c(\lambda) = \lambda^2 M + \lambda (D + G - BF^T) + (K - BG^T) \]

are replaced by “suitably” chosen ones leaving the rest unchanged.

In several recent papers [7, 8, 10, 17], the partial pole-placement for the damped nongyroscopic systems (\( G = 0 \)) has been considered and a “novel” approach, called the “direct partial-modal” approach has been developed.

This approach has several distinct features. First, the solution requires only those few eigenvalues that need to be reassigned and the corresponding eigenvectors.

Second, the problem is solved completely in the second-order setting; that is no transformation to a first-order system is invoked; thus making it possible to preserve the exploitable structures such as the sparsity, definiteness, bandness, etc., very often offered by many practical problems.

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Furthermore, mathematical results are proven that guarantee that there will be no spill-over during the process; that is, the eigenvalues and eigenvectors that are not changed will remain invariant.

In this paper, we extend our study to the gyroscopic systems but neglecting damping. Our study this time goes beyond the matrix second-order system and extends to the operator systems of which the former is just a discretized approximation. We present our results here for the single-input case only. Generalizations of these results to the multi-input case are presently being studied and will be reported elsewhere [9].

Specifically, we solve the single-input partial pole placement problem for the control operator systems of the form:

\[ Mu_{tt}(t, x) + Gu_{t}(t, x) + Ku(t, x) = b(x) h(t) \]

where \( M \) and \( K \) are self-adjoint positive definite operators, \( G \) is skew-symmetric, \( b(x) \) is a control function, and \( h(t) = (f(x), u_{t}(t, x)) + (g(x), u(t, x)) \). This problem arises, for example, in regulating the vibratory effects of small oscillations of a taut string, rotating about its \( x \)-axis with constant angular velocity; of small oscillations of a uniform string traveling with constant velocity \( \gamma \) over the fixed supports, etc. In the special case, when \( M, G \), and \( K \) are matrices and \( b \) is an input control vector, we obtain the solution of the single-input partial pole placement problem for a matrix second-order system of the form (1).

Our solution technique again is “partial modal” and has the same distinctive features as of our solution of the single-input partial eigenvalue assignment for the nongyroscopic matrix second systems, described above.

The paper is organized as follows.

In Section 2, we define and describe gyroscopic operator systems and give a few examples of the gyroscopic systems.

In Section 3, we establish two important properties of the eigenvalues and eigenvectors of a gyroscopic operator pencil. In Section 4, we give a mathematical formulation of our problem and establish the existence and uniqueness of the solution.

In Section 5, we demonstrate the effectiveness of our solution using a few practical examples.

The papers concludes with some remarks on the possible practical significance of our presented approach.

## 2 Gyroscopic Systems

Denote the domain of the spatial coordinates by \( X \), and the relevant Hilbert space by \( V \subset \{ f(x) : X \to \mathbb{C} \} \). The scalar product \( \langle \cdot , \cdot \rangle \) is such that \( \langle \alpha v, \beta w \rangle = \langle \beta w, \alpha v \rangle = \overline{\alpha} \beta \langle v, w \rangle \) for all \( v, w \in V \) and \( \alpha, \beta \in \mathbb{C} \), where bar denotes complex conjugation. Homogeneous boundary conditions are taken into account by restricting the displacement function \( u(t, x) \in C^2(\mathbb{R}, V) \), i.e. \( u(t, x) \) is twice continuously differentiable function from the real numbers to the Hilbert space \( V \).

Let the dual of \( V \) be denoted by \( V' \), and let a linear operator \( A \in V' \) be such that

\[ \langle Av, w \rangle = \langle v, Aw \rangle, \quad \langle Av, v \rangle > 0, \]

for all \( v, w \in V \), \( ||v|| \neq 0 \). Then \( A \) is said to be a self-adjoint positive definite operator.

A linear operator \( B \in V' \) which satisfies

\[ \langle Bv, w \rangle = -\langle v, Bw \rangle, \]

for all \( v, w \in V \), is called a skew-symmetric operator.

We now present three practical systems that fit into the general framework of

\[ Mu_{tt} + Gu_{t} + Ku = 0, \]

where \( M \) and \( K \) are self-adjoint positive definite operators and \( G \) is skew-symmetric. The operator pencil associated with (4) is

\[ P_{G}(\lambda) = \lambda^{2}M + \lambda G + K. \]

### 2.1 The Oscillations of a Frame

Consider the oscillations of a vibrating system, attached to a rigid frame. Suppose that the frame rotates with a constant angular velocity. Then the free infinitesimal oscillations of the system of partial differential equations:

\[ \ddot{M}x(t) + G\dot{x}(t) + Kx(t) = 0, \]

where \( M \) and \( K \) are \( n \times n \) symmetric positive definite mass and stiffness matrices and \( G = -G^{T} \) is an \( n \times n \) skew-symmetric matrix.

Clearly with \( X = \{1, 2, \ldots, n\} \), \( V = \mathbb{C}^{n} \) and the scalar product \( \langle v, w \rangle = \overline{w}^{T}v \), the system (6) falls into this framework.

### 2.2 Small Oscillations of a Taut String

Consider the motion of a taut string, rotating about its \( x \)-axis with constant angular velocity \( \omega_{1} \). It will be shown in Section 5 that the small oscillations of such a string are governed by the system of partial differential equations:

\[ \begin{cases} \psi_{tt}(t, x) - 2\omega_{1}\xi_{t}(t, x) - \omega_{1}^{2}\psi(t, x) - c^{2}(x)\psi_{xx}(t, x) = 0 \\ \xi_{tt}(t, x) + 2\omega_{1}\psi_{t}(t, x) - \omega_{1}^{2}\xi(t, x) - c^{2}(x)\xi_{xx}(t, x) = 0 \end{cases} \]

Let

\[ X = \{1, 2\} \times [0, 1], \]

\[ V = \{ v = [v(1, x), v(2, x)]^{T} : v(i, x) \in \mathbb{C}^{2}([0, 1]) \}; \]

\[ v(i, 0) = v(i, 1) = 0 \text{ for } i = 1, 2 \}, \]

and

\[ \langle v, w \rangle = \int_{0}^{L} \sum_{i=1}^{2} \int_{0}^{1} v(i, x)w(i, x) \, dx \quad \text{for any } v, w \in V. \]
Then we find that the system can be written in the form 4 where

\[ M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \]

and

\[ K = \begin{bmatrix} -c^2(x) \frac{\partial^2}{\partial x^2} - \omega_1^2 & 0 \\ 0 & -c^2(x) \frac{\partial^2}{\partial x^2} - \omega_1^2 \end{bmatrix}. \]

For a sufficiently small \( \omega \) the operator \( K \) is positive definite.

2.3 The Oscillations of a Uniform String

Consider motion of a uniform string traveling with constant velocity \( \gamma \) over two fixed supports at \( x = 0 \) and \( x = 1 \). It can be shown (see [18]) that small oscillations of such a string are governed by the partial differential equations

\[ u_{tt}(t, x) + 2\gamma u_{xx}(t, x) + (\gamma^2 - c^2)u_{xx}(t, x) = 0 \quad (7) \]

with the boundary conditions:

\[ u(0, t) = u(t, t) = 0. \quad (8) \]

We now show that the system (7) is gyroscopic. Selecting

\[ \mathbf{X} = [0, L], \quad \mathbf{V} = \{v(x) \in C^2([0, 1]) : v(0) = v(1) = 0\} \]

\[ (v, w) = \int_0^L v(x)w(x) \, dx \quad \text{for any} \quad v, w \in \mathbf{V}, \]

\[ Mv = v, \quad Gv = 2\gamma \frac{\partial v}{\partial x}, \quad Kv = (\gamma^2 - c^2) \frac{\partial^2 v}{\partial x^2}, \]

gives

\[ (Mv, w) = \int_0^L w(x)v(x) \, dx = (Mw, v) = (v, Mw), \]

\[ (Mv, v) = \int_0^L |v(x)|^2 \, dx \geq 0. \]

Integrating by parts yields

\[ (Kv, w) = -\int_0^L (\gamma^2 - c^2)u''(x)v(x) \, dx = (v, Kv), \]

and

\[ (Kv, v) = \int_0^L (\gamma^2 - c^2)|v'(x)|^2 \, dx > 0, \]

in view of the boundary conditions (8). Since \( v'(x) \) does not vanish identically, \( K \) is a self–adjoint positive definite operator. Another integration gives

\[ (Gv, w) = -\int_0^L 2\gamma u''(x)v(x) \, dx = -(v, Gw), \]

and hence the traveling string is a gyroscopic system.

3 Some Spectral Properties of Gyroscopic Systems

From now on \( M \) and \( K \) are considered to be self–adjoint positive definite operators and \( G \) denotes a gyroscopic operator.

With separation of variables \( u(t, x) = e^{\lambda t}v(x) \) the system (5) reduces to the eigenvalue problem

\[ (\lambda^2 M + \lambda G + K)v(x) = 0 \quad \text{and} \quad v(x) \in \mathbf{V}. \quad (9) \]

The scalars \( \lambda \) and the corresponding functions \( v_j(x) \) which non–trivially solves (9) are called eigenvalues and eigenfunctions of the operator pencil \( PG(\lambda) \), respectively.

The spectral properties of gyroscopic systems are well understood [2, 11, 12, 15]. For the sake of completeness, however, we redevelop here some results needed for our analysis.

**Proposition 1** The eigenvalues of (9) are purely imaginary.

**Proof.** Let \( \lambda \) and \( v(x) \) be an eigenpair of (9). The scalar product of (9) with \( \lambda v(x) \in \mathbf{V} \) gives

\[ -(\lambda v, \lambda Gv) = (\lambda v, \lambda^2 Mv + (\lambda v, Kv), \quad (10) \]

\[ -(\lambda Gv, \lambda v) = (\lambda^2 Mv, \lambda v) + (Kv, \lambda v), \quad (11) \]

and by (2) and (3) we have

\[ (\lambda v, \lambda Gv) = (\lambda^2 v, \lambda Mv) + (v, \lambda Kv). \quad (12) \]

Adding (10) and (12) gives

\[ 0 = (\lambda + \lambda)(\lambda Mv) + (v, K v) \]. \quad (13) \]

Since \( (\lambda v, M \lambda v) > 0 \) and \( (v, K v) > 0 \) we necessarily have \( \lambda + \lambda = 0 \), i.e. \( \lambda \) is purely imaginary.

A first order realization of (9) is given by

\[ \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} v \\ \lambda v \end{bmatrix} \lambda = \begin{bmatrix} 0 & K \\ -K & -G \end{bmatrix} \begin{bmatrix} v \\ \lambda v \end{bmatrix}. \quad (14) \]

Denote \( \hat{\lambda} = i\lambda \) and \( \phi + i\eta = [v, \lambda v]^T \). Then, since \( \lambda \) is purely imaginary the system (9) has the following symmetric realization

\[ \hat{\lambda} \begin{bmatrix} K & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \eta \\ \phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & K \\ 0 & -K & -G \\ 0 & -K & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \phi \end{bmatrix}, \quad (15) \]

where the left-hand-side matrix is positive definite. If \[ \{\hat{\lambda}, \begin{bmatrix} \eta \\ \phi \end{bmatrix}\} \] is an eigenpair of (15) then (15) also has the eigenpairs \[ \{\hat{\lambda}, \begin{bmatrix} \eta + \phi \\ \eta \end{bmatrix}\}, \quad \{-\hat{\lambda}, \begin{bmatrix} -\eta \\ \phi \end{bmatrix}\} \] and
Theorem 1 \cite{5, p. 507}, the following result holds for gyroscopic systems.

\[ \begin{pmatrix} -\lambda, \eta + \phi \\ -\eta - \phi \end{pmatrix} \] . It follows therefore that the real eigenpairs of (15) determine the generally complex eigenpairs of (9) uniquely, and vice versa.

We now introduce the Hilbert space \( \mathbf{W} = \mathbf{V} \times \mathbf{V} \) associated with (14), such that for each \( [v_1, v_2]^T, [w_1, w_2]^T \in \mathbf{W} \) the scalar product

\[ \langle [v_1, v_2]^T, [w_1, w_2]^T \rangle = (Kv_1, w_1) + (Mv_2, w_2). \] (16)

For each eigenpair \( \{\lambda_j, v_j\} \) of (9) we thus have the counterpart \( \{\lambda_j, \hat{v}_j\} \) of the first order realization (14), where

\[ \hat{v}_j = \begin{bmatrix} v_j \\ \lambda_j v_j \end{bmatrix}, \] (17)

and conclude that

**Proposition 2** The eigenfunctions (17) are complete in \( \mathbf{W} \).

In analogy to the well known biorthogonal relations for the symmetric generalized eigenvalue problem see, e.g. \cite[p. 507]{5}, the following result holds for gyroscopic systems.

**Theorem 1** Let \( v_j, v_k \) be two eigenfunctions of (9) associated with two eigenvalues \( \lambda_j \neq \lambda_k \). Then

\[ \langle \hat{v}_k, \hat{v}_k \rangle = (\lambda_k (Mv_k), (Kv_k, v_j)) + (Kv_k, v_j) = 0. \] (18)

**Proof.** Similar to (10)-(11) we write

\[ -\langle \lambda_k v_k, \lambda_j Gv_j \rangle = (\lambda_k (Gv_k), (\lambda_j (Gv_k), v_j)) + (\lambda_k v_k, K v_j), \] (19)

and

\[ -\langle \lambda_k Gv_k, \lambda_j v_j \rangle = (\lambda_k^2 (v_k), (\lambda_j v_j, v_j)) + (Kv_k, v_j) v_j. \] (20)

Then

\[ (\lambda_k v_k, \lambda_j Gv_j) = (\lambda_k^2 v_k, \lambda_j (Mv_j)) + (v_k, \lambda_j K v_j), \] (21)

by virtue of (2) and (3). Adding (19) and (21) gives

\[ 0 = \left( \sum_k + \lambda_j \right) \left( (\lambda_k v_k, (\lambda_k^2 Mv_j)) + (v_k, (Kv_j, v_j)) \right), \] (22)

and it thus follow from Proposition 1 that (18) holds.

If \( \lambda_p \) is a semi-simple eigenvalue of multiplicity \( q \) then we may choose \( q \) linearly independent eigenfunctions such that (18) holds. This can be done, for example, by using Gram–Schmidt orthogonalization.

4 Partial Pole-Assignment for Single-Input Gyroscopic Systems

Let \( \{\mu_1, \ldots, \mu_m\} \) be a prescribed self-conjugate set of complex numbers. Define

\[ \Omega = \{\{\mu_j\}_{i=1}^m, \{\lambda_j\}_{j=m+1}^\infty\}, \] (23)

where \( \lambda_j, j = m + 1, \ldots, \) are the eigenvalues of the open-loop operator pencil \( P_{0}(\lambda) = \lambda^2 M + \lambda G + K \) associated with the operator system (9).

We consider here the following problem

**Problem 1.** Suppose \( M, G, K, b(x) \) and a complex set \( \{\mu_j\}_{j=1}^m \) are given. Find two functions, \( f(x) \) and \( g(x) \), such that each element of \( \Omega \), defined by (23), is an eigenvalue of the modified system

\[ M v_t + G v_t + K v = (f, v_t) b + (g, v) b. \] (24)

Separation of variables \( v(t, x) = e^{nt} w(x) \) gives

\[ (\mu^2 M + \mu G + K) w = (f, w) b + (g, w) b, w(x) \in \mathbf{V}. \] (25)

The pair \( \{\mu_j, w_j(x)\} \) which non-trivially solves (25) is called an eigenpair of the modified system (24).

**Theorem 2** Suppose that the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of (5) are distinct, \( \mu_1 \notin \{\lambda_k\}_{k=1}^m \), and \( (b, v_j) \neq 0 \) for \( j = 1, \ldots, m \). Then the solution to Problem 1 is given by

\[ f(x) = \sum_{j=1}^m \beta_j \lambda_j M v_j, \quad g(x) = \sum_{j=1}^m \beta_j K v_j, \] (26)

where

\[ \beta_j = \frac{1}{(b, \lambda_j v_j)} \prod_{i=1}^m (\lambda_i - \lambda_j). \] (27)

**Proof.** We first show that with (26) each \( \{\lambda_k, v_k\} \), for \( k > m \), is an eigenpair of (25) when \( \beta_1, \ldots, \beta_m \) are arbitrarily chosen. Indeed, substituting (26) in (25) gives

\[ \lambda_k^2 M v_k + \lambda_k G v_k + K v_k, \] (28)

by virtue of the orthogonal relation (18).

Let \( w_k(x) \in \mathbf{V} \) be the solution of

\[ \mu_k^2 M w_k(x) + \mu_k G w_k(x) + K w_k(x) = b(x). \] (29)

Such a solution exists by the Fredholm alternative, since \( \mu_k \) is not in the spectrum of (5).

We now show that with \( \beta_1, \beta_2, \ldots, \beta_m \) chosen according to (27), each \( \{\mu_j, w_j\} \) is an eigenpair of (25). Similar to (19) and (20) we obtain for \( j = 1, 2, \ldots, m \)

\[ -(\mu_k w_k, \lambda_j G v_j) = (\mu_k w_k, \lambda_j^2 M v_j) + (\mu_k w_k, K v_j) \] (29)

and

\[ -(\mu_k G w_k, \lambda_j v_j) = (\mu_k^2 M w_k, \lambda_j v_j) + (K w_k, \lambda_j v_j) - (b, \lambda_j v_j). \] (30)
Adding (29) and (30) gives
\[ (b, \lambda_j v_j) = (\overline{\mu_k + \lambda_j}) (\mu_k M w_k, \lambda_j v_j) + (K w_k, v_j) \]  
for some constants \( \beta_k \), by virtue of the completeness of the eigenfunctions. It thus follows that
\[ g = \sum_{k \geq 1} \beta_k Kv_k, \quad f = \sum_{k \geq 1} \beta_k \lambda_k M v_k. \]  
It will now be shown that \( \beta_k = 0 \) for all \( k > m \). Let \( \{ \lambda_j, v_j \} \) and \( \{ \lambda_j, w_j \} \) be normalized eigenpairs of (9) and (25) respectively, in the sense that \( ||v_j|| = ||w_j|| = 1 \). Then by (9) and (25) we have
\[ (w_j, \lambda_j^2 M v_j) + (w_j, \lambda_j G v_j) + (w_j, K v_j) = 0 \]  
and
\[ (\lambda_j^2 M v_j, v_j) + (\lambda_j G w_j, v_j) + (K w_j, v_j) = (b, v_j) ((f, \lambda_j w_j) + (g, w_j)) \]  
Subtracting (37) from (38) gives
\[ 0 = (b, v_j) ((f, \lambda_j w_j) + (g, w_j)) \]  
by virtue of (2)-(3) and Proposition 1. Since \( (b, v_j) \neq 0 \) it follows from (25) and (39) that
\[ \lambda_j^2 M v_j + \lambda_j G w_j + K w_j = 0. \]
Hence \( \{ \lambda_j, w_j \} \) is also an eigenpair of (5). Suppose that \( v_j \) and \( w_j \) are linearly independent. Then \( \tilde{w} = (b, v_j) w_j - (b, w_j) v_j \) is an eigenfunction of (5). But \( (b, \tilde{w}) = 0 \) contradicts our assumption that \( b \) is not orthogonal to the eigenfunctions of (5). It thus concluded that \( v_j = w_j \).  
Since \( b(x) \) is not identically zero it follows from (9) and (25) that
\[ (f, \lambda_j v_j) + (g, v_j) = 0 \quad \text{for all} \quad j > m. \]  
Substituting (36) in (40) yields
\[ \sum_{k \geq 1} \beta_k \lambda_k M v_k, \lambda_j v_j) + \sum_{k \geq 1} \beta_k K v_k, v_j = (b, v_j) ((f, \lambda_j w_j) + (g, w_j)) \]  
and
\[ \sum_{k \geq 1} \beta_k ((\lambda_k M v_k, \lambda_j v_j) + (K v_k, v_j)) = 0 \]  
in view of (18). The positive definiteness of \( M \) and \( K \) implies that \( \beta_j = 0 \) for all \( j > m \). Hence \( f \) and \( g \) have the form (26). The proof is completed by the uniqueness of \( \beta_1, \beta_2, \ldots, \beta_m \), established in the proof of Theorem 2.  
Moreover, by the completeness of \( \{ \tilde{v}_j \} \) we have
\[ \tilde{w}_k = \begin{bmatrix} w_k \\ \mu_k w_k \end{bmatrix} = \sum_{j \geq 1} d_{kj} \tilde{v}_j, \quad k = 1, 2, \ldots, m, \]  
and by (18)
\[ < \tilde{w}_k, \tilde{v}_j > = d_{kj} < \tilde{v}_j, \tilde{v}_j > \]  
Hence (31) can be written in the form
\[ < \tilde{w}_k, \tilde{v}_j > = \frac{(b, \lambda_j v_j)}{(\overline{\mu_k + \lambda_j})}. \]
and it thus follows that
\[ d_{kj} = \frac{(b, \lambda_j v_j)}{(p_k + \lambda_j)} < v_j, \dot{v}_j > \]
for all \( 1 \leq k, j \leq m \). Using (34) we obtain
\[
\begin{bmatrix}
\dot{v}_1 \\
\vdots \\
\dot{v}_p
\end{bmatrix} = C^{-1}
\begin{bmatrix}
\leq \dot{\xi}_1, \dot{\xi}_1 > (\dot{w}_1 - \sum_{i>m} d_{1i} \dot{v}_i) \\
\vdots \\
\leq \dot{\xi}_m, \dot{\xi}_m > (\dot{w}_m - \sum_{i>m} d_{mi} \dot{v}_i)
\end{bmatrix}.
\]
The eigenfunctions of Problem 1 with \( f \) and \( g \) as in (26)-(27) are thus complete in \( V \). Therefore, it is concluded that the spectrum of (25) with (26) and (27) is precisely the set \( \Omega \).

## 5 Modeling and Examples

Let \( X-Y-Z \) be a stationary coordinate system, and let \( x-y-z \) be another coordinate systems of origin \( O \) which moves in a general space motion with respect to the stationary coordinates, as shown in Fig. 1.

Denote the angular velocity and angular acceleration of \( x-y-z \) by \( \omega \) and \( \ddot{\omega} \). Let \( r \) and \( r_0 \) be the position vectors of a particle \( P \) and the origin of \( O \) of \( x-y-z \), respectively. Denote the position of \( P \) with respect to the \( x-y-z \) coordinates by the vector \( u \). Then in Newtonian dynamics we have
\[
\begin{align*}
\dot{r} &= r_0 + u, \\
\ddot{r} &= \ddot{r}_0 + \ddot{u} + \omega \times u
\end{align*}
\]
and
\[
\dddot{r} = \dddot{r}_0 + \dddot{u} + 2\omega \times \dot{u} + \omega \times (\omega \times u) + \ddot{\omega} \times u.
\]

where \( \dddot{r} \) and \( \dddot{r} \) are the velocity and acceleration of \( P \) as observed from the stationary coordinates, \( \dot{u} \) and \( \ddot{u} \) are velocity and acceleration with respect to an observer fixed to \( x-y-z \). Let \( R \) be the resultant of all forces applied to the particle \( P \). Then by Newton’s second law
\[
R = m\dddot{u},
\]
where \( m \) is the mass of \( P \).

**Spatial oscillations of a particle.** Let the particle of mass \( m \) be connected to a ring of radius \( \alpha \) via two springs of constant \( k \) and free length \( \beta \). Suppose that the ring is rotating with constant angular velocity \( \omega = (\omega_1, \omega_2, \omega_3)^T \). Let the moving frame \( x-y-z \) be attached to the ring, with its origin \( O \) at the center of the ring, and with its axis \( x \) aligned with \( \overline{OP} \), as shown in Fig. 2. Then for small oscillations about the equilibrium position, Hook’s law gives
\[
R = \begin{bmatrix} -2\kappa & -2\kappa \gamma & 0 \\ -2\kappa \gamma & -2\kappa \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},
\]
where \( \gamma = (1 - \beta/\alpha) \), and \( u_1, u_2, u_3 \) are the displacements of the mass in the \( x-, y-, z- \) direction. Since \( O \) is a fixed point of rotation, \( \dddot{r}_0 = 0 \), and we have by (44)
\[
\dddot{r} = \dddot{u} + 2\omega \times \dot{u} + \omega \times (\omega \times u)
\]
\[
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dddot{u}_1 \\ \dddot{u}_2 \\ \dddot{u}_3 \end{bmatrix}
\]
\[
+ \begin{bmatrix} 2\omega_3 & 2\omega_2 & 0 \\ 0 & 2\omega_3 & 2\omega_1 \\ 2\omega_2 & 0 & 2\omega_1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}
\]
Consider the traveling string modeled by (7) and (8), are functions. The problem is thus considered as a non-defective eigenvalue problem where $\psi$ and $\xi$ are the displacements of the element of the string in the $y$– and $z$–direction, respectively. Since the origin of $x − y − z$ is stationary, (44) gives

$$
\ddot{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ \xi_{tt} \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} -2\omega_2 & 0 \\ 0 & -\omega_3^2 \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix} = 0,
$$

Hence by Newton’s second law

$$
\ddot{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ 0 \end{bmatrix} + \begin{bmatrix} -2\omega_2 & 0 \\ 0 & -\omega_3^2 \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix} = 0,
$$

where $c^2(x) = T/\rho(x)$. The system has the following eigenpairs

$$
\left\{ \lambda_k, e^{\pm \frac{2\pi x}{T} - \frac{i}{1}} \right\} = \left\{ -\lambda_k, e^{\pm \frac{2\pi x}{T} + \frac{i}{1}} \right\},
$$

where $\lambda_k = i(1 + 2\pi k)$ for all integers $k \neq 0$. Note that the eigenvalues $\pm \lambda_k$ have multiplicity two. However, for each double eigenvalue there exist two linearly independent eigenfunctions. The problem is thus non-defective.

To demonstrate the partial spectral modification we chose $\omega_1 = 1$ and $c^2 \equiv 4$. We wish to assign the complex conjugate pair \{$(1 + 2\pi i), -i(1 + 2\pi i)$\} to \{-1 + i, -1 - i\}. Using (26) and (27) we find

$$
f = \begin{bmatrix} -2.0000 \\ 0.1217 \\ 3.9485 \end{bmatrix}, \quad g = \begin{bmatrix} -16.7625 \\ 14.9362 \\ -11.7818 \end{bmatrix}.
$$

Small oscillations of a taut rotating string. Consider the taut string of density $\rho(x)$, stretched with tension $T$, attached to a rotating frame, as shown in Fig. 3. Let the angular velocity of the frame be $\omega = (\omega_1, 0, 0)^T$. For an infinitesimal element of the string vibrating in the $y−z$ plane the Newton’s second law gives

$$
R = \begin{bmatrix} T\psi_{yy} \\ T\psi_{zz} \end{bmatrix}.
$$

where $\psi$ and $\xi$ are the displacements of the element of the string in the $y$– and $z$–direction, respectively. Since the origin of $x − y − z$ is stationary, (44) gives

$$
\ddot{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ \xi_{tt} \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} -2\omega_2 & 0 \\ 0 & -\omega_3^2 \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix} = 0,
$$

Hence by Newton’s second law

$$
\ddot{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ 0 \end{bmatrix} + \begin{bmatrix} -2\omega_2 & 0 \\ 0 & -\omega_3^2 \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix} = 0,
$$

where $c^2(x) = T/\rho(x)$. The system has the following eigenpairs

$$
\left\{ \lambda_k, e^{\pm \frac{2\pi x}{T} - \frac{i}{1}} \right\} = \left\{ -\lambda_k, e^{\pm \frac{2\pi x}{T} + \frac{i}{1}} \right\},
$$

where $\lambda_k = i(1 + 2\pi k)$ for all integers $k \neq 0$. Note that the eigenvalues $\pm \lambda_k$ have multiplicity two. However, for each double eigenvalue there exist two linearly independent eigenfunctions. The problem is thus non-defective.

To demonstrate the partial spectral modification we chose $\omega_1 = 1$ and $c^2 \equiv 4$. We wish to assign the complex conjugate pair \{$(1 + 2\pi i), -i(1 + 2\pi i)$\} to \{-1 + i, -1 - i\}. Using (26) and (27) we obtain

$$
f = \frac{\pi}{4 + 8\pi} \begin{bmatrix} (3 - 4\pi^2) \cos(\pi x) \\ (1 + 8\pi + 4\pi^2) \cos(\pi x) \end{bmatrix} + \frac{\pi}{4 + 8\pi} \begin{bmatrix} (1 + 8\pi + 4\pi^2) \sin(\pi x) \\ -(3 - 4\pi^2) \sin(\pi x) \end{bmatrix},
$$

$$
g = \frac{\pi - 2\pi^2}{4 + 8\pi} \begin{bmatrix} (1 + 8\pi + 4\pi^2) \cos(\pi x) \\ (4\pi^2 - 3) \cos(\pi x) \end{bmatrix} + \frac{\pi - 2\pi^2}{4 + 8\pi} \begin{bmatrix} (3 - 4\pi^2) \sin(\pi x) \\ (1 + 8\pi + 4\pi^2) \sin(\pi x) \end{bmatrix}.
$$

It may be confirmed by direct substitution that despite the applied force, all other eigenpairs remain unchanged.

Small oscillations of a traveling string. With $c = 1, \gamma = 0.5$ and $b = 1$ the eigenpairs of the traveling string, modeled by (7) and (8), are

$$
\left\{ \lambda_k, e^{\pm \frac{2\pi x}{T} - e^{2\lambda_k x}} \right\},
$$

where $\lambda_k = i(1 + 2\pi k)$ for all integers $k \neq 0$. Note that the eigenvalues $\pm \lambda_k$ have multiplicity two. However, for each double eigenvalue there exist two linearly independent eigenfunctions. The problem is thus non-defective.

To demonstrate the partial spectral modification we chose $\omega_1 = 1$ and $c^2 \equiv 4$. We wish to assign the complex conjugate pair \{$(1 + 2\pi i), -i(1 + 2\pi i)$\} to \{-1 + i, -1 - i\}. Using (26) and (27) we obtain

$$
f = \frac{\pi}{4 + 8\pi} \begin{bmatrix} (3 - 4\pi^2) \cos(\pi x) \\ (1 + 8\pi + 4\pi^2) \cos(\pi x) \end{bmatrix} + \frac{\pi}{4 + 8\pi} \begin{bmatrix} (1 + 8\pi + 4\pi^2) \sin(\pi x) \\ -(3 - 4\pi^2) \sin(\pi x) \end{bmatrix},
$$

$$
g = \frac{\pi - 2\pi^2}{4 + 8\pi} \begin{bmatrix} (1 + 8\pi + 4\pi^2) \cos(\pi x) \\ (4\pi^2 - 3) \cos(\pi x) \end{bmatrix} + \frac{\pi - 2\pi^2}{4 + 8\pi} \begin{bmatrix} (3 - 4\pi^2) \sin(\pi x) \\ (1 + 8\pi + 4\pi^2) \sin(\pi x) \end{bmatrix}.
$$

It may be confirmed by direct substitution that despite the applied force, all other eigenpairs remain unchanged.

Small oscillations of a traveling string. With $c = 1, \gamma = 0.5$ and $b = 1$ the eigenpairs of the traveling string, modeled by (7) and (8), are

$$
\left\{ \lambda_k, e^{\pm \frac{2\pi x}{T} - e^{2\lambda_k x}} \right\},
$$

where $\lambda_k = i(1 + 2\pi k)$ for all integers $k \neq 0$. Note that the eigenvalues $\pm \lambda_k$ have multiplicity two. However, for each double eigenvalue there exist two linearly independent eigenfunctions. The problem is thus non-defective.
where \( \lambda_k = \frac{3}{4}\pi i(\pm 1 + 4k) \) for all integers \( k \). The functions

\[
\begin{align*}
f(x) &= -\frac{1}{32} \left( (9\pi^2 + 24\pi - 32) \cos \left( \frac{\pi x}{2} \right) \right. \\
&\quad \left. + (9\pi^2 - 24\pi - 32) \sin \left( \frac{\pi x}{2} \right) \right) \sin(\pi x) \\
g(x) &= \frac{\pi}{256} \left( 9(9\pi^2 + 24\pi - 32) \cos \left( \frac{3\pi x}{2} \right) \right. \\
&\quad \left. + (32 - 24\pi - 9\pi^2) \cos \left( \frac{\pi x}{2} \right) \\
&\quad \left. + (9\pi^2 - 24\pi - 32) \left( \cos \left( \frac{\pi x}{2} \right) + 9 \sin \left( \frac{3\pi x}{2} \right) \right) \right)
\end{align*}
\]

are the eigenfunctions of the string (in a weak solution sense) and its complex conjugate, corresponding to the eigenvalues \( -1 + i \) and \(-1 - i\), respectively. All other eigenpairs are unaltered by the applied force.

\[ f(x) = \sum_{j=1}^{m} \beta_j \lambda_j M v_j , \quad g(x) = -\sum_{j=1}^{m} \beta_j K v_j , \]

\[ \beta_j = \frac{1}{(b, \lambda_j v_j)} \prod_{i=1 \atop i \neq j}^{m} (\lambda_i - \lambda_j) . \]

This result has been established for the finite dimensional case in [8].

### References


