An optimization problem and
Nevanlinna-Pick expansion:
Extended abstract.

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1 A positive expansion problem

In this talk we will use a modification of a classical optimization problem in prediction theory [7, 8] to derive a maximal solution for the Nevanlinna-Pick expansion problem in [3, 4, 9] for each point in the open unit disc. This optimization problem is also used to show that the maximal solution is unique. State space formulas are given. Finally, it is shown that these maximal solutions will convergence to the appropriate outer spectral factor as the data set becomes large.

To introduce the Nevanlinna-Pick expansion problem, let $T$ on $\ell_+^2(U)$ be the strictly positive Toeplitz operator matrix given by

$$
T = \begin{bmatrix}
R_0 & R_1 & R_2 & \cdots \\
R_{-1} & R_0 & R_1 & \cdots \\
R_{-2} & R_{-1} & R_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \text{ on } \ell_+^2(U).
$$

(1.1)

Now let $\{C, A\}$ be a stable pair, where $A$ is an operator on $X$ and $C$ is an operator mapping $X$ into $U$. By stable we mean that the spectrum of $A$ is contained in the open unit disc $\mathbb{D}$. Moreover, we assume that $C$ is onto. Let $W$ be the observability operator mapping $X$ into $\ell_+^2(U)$ defined by

$$
W = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
\end{bmatrix}.
$$

(1.2)

Throughout we assume that the pair $\{C, A\}$ is observable. In other words, we assume that $W$ is left invertible, or equivalently, $W^*W$ is invertible. Hence $\Lambda = W^*TW$ is a strictly positive operator on $X$.

The operator $\Lambda = W^*TW$ is a solution to a Lyapunov equation of the form

$$
\Lambda = A^*A + C^*\tilde{C} + \tilde{C}^*C.
$$

(1.3)

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Here $\tilde{C}$ is an operator from $\mathcal{X}$ into $\mathcal{U}$. This naturally leads to a Nevanlinna-Pick expansion problem. The data set is a triple of operators $\{A, C, \Lambda\}$ where $\{C, A\}$ is a stable, observable pair and $C$ is onto. Moreover, $\Lambda$ is a strictly positive operator on $\mathcal{X}$ satisfying a Lyapunov equation of the form (1.3). Then our Nevanlinna-Pick expansion problem is to find the set of all strictly positive Toeplitz operators $T$ on $\ell^2_+ (\mathcal{U})$ satisfying $\Lambda = W^* T W$. The set of all solutions to this problem is given in [3, 4, 9]. In this talk we will use some techniques from classical prediction theory [7, 8] to present an elementary derivation of a special solution to this expansion problem for each fixed $\alpha$ in the open unit disc $\mathbb{D}$. For $\alpha = 0$ this solution turns out to be the central solution to the Nevanlinna-Pick expansion problem presented in [3, 4, 9]. The optimization problem arising in classical prediction theory will also be used to show that the maximal solution is unique. Finally, it is noted that we do not need to obtain $\tilde{C}$ to compute a solution to our Nevanlinna-Pick expansion problem. All we need to know is that $\Lambda$ is a solution to a Lyapunov equation of the form (1.3).

This expansion problem encompasses the classical Carathéodory interpolation problem. To see this assume that $A$ is the upper shift on $\mathcal{X} = \bigoplus^n_1 \mathcal{U}$ given by

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \quad (1.4)$$

Notice that the state space $\mathcal{X}$ is simply $n$ orthogonal copies of $\mathcal{U}$. In this setting the observability operator $W$ in (1.2) is given by

$$W = \begin{bmatrix} I \\ 0 \end{bmatrix} : \bigoplus^n_1 \mathcal{U} \to \ell^2_+ (\mathcal{U}) .$$

In other words, $W$ embeds $\mathcal{X} = \bigoplus^n_1 \mathcal{U}$ into the first $n$ components of $\ell^2_+ (\mathcal{U})$. Now assume that $T$ is the strictly positive Toeplitz operator given in (1.1). Then $\Lambda = W^* T W$ is the strictly positive $n \times n$ Toeplitz matrix contained in the upper left hand corner of $T$, that is,

$$\Lambda = W^* TW = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-1} \\ R_{-1} & R_0 & R_1 & \cdots & R_{n-2} \\ R_{-2} & R_{-1} & R_0 & \cdots & R_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{1-n} & R_{2-n} & R_{3-n} & \cdots & R_0 \end{bmatrix} . \quad (1.5)$$

Now assume that $\Lambda$ is a strictly positive Toeplitz matrix of the form (1.5). Then $\Lambda$ is a solution to a Lyapunov equation of the form in (1.3). Therefore $\{A, C, \Lambda\}$ with $\{C, A\}$ in (1.4) and $\Lambda$ in (1.5) strictly positive is a data set for our Nevanlinna-Pick expansion problem. In this setting, our Nevanlinna-Pick expansion problem is to find the set of all strictly positive Toeplitz operators $T$ on $\ell^2_+ (\mathcal{U})$ such that $\Lambda$ is contained in the $n \times n$ upper left hand corner of $T$. This is precisely the classical Carathéodory interpolation problem.
To introduce our solution to the Nevanlinna-Pick expansion problem, we need some additional notation. Let \( \Theta \) be a function in \( H^\infty(U,U) \). (Here \( H^\infty(U,U) \) is the Hardy space consisting of the set of all uniformly bounded analytic functions in the open unit disc whose values are bounded operators on \( U \).) Then \( T_\Theta \) is the lower triangular Toeplitz operator on \( \ell_+^2(U) \) defined by
\[
T_\Theta = \begin{bmatrix}
\Theta_0 & 0 & 0 & \cdots \\
\Theta_1 & \Theta_0 & 0 & \cdots \\
\Theta_2 & \Theta_1 & \Theta_0 & \cdots \\
& & & & \ddots
\end{bmatrix}
on \ell_+^2(U)
\] (1.6)
where \( \Theta(\lambda) = \sum_0^\infty \Theta_n \lambda^n \) is the Taylor series expansion for \( \Theta \). We say that \( \Theta \) is an invertible outer function if both \( \Theta \) and \( \Theta^{-1} \) are functions in \( H^\infty(U,U) \). We say that \( \Theta \) is an outer spectral factor for the Toeplitz operator \( T \) in (1.1) if \( \Theta \) is an outer function in \( H^\infty(U,U) \) satisfying \( T = T_\Theta^*T_\Theta \).

Now assume that \( T \) is a solution to the Nevanlinna-Pick expansion problem for the data set \( \{A, C, \Lambda\} \). In other words, assume that \( T \) is a strictly positive Toeplitz operator such that \( \Lambda = \Lambda^*T\Lambda \). Then \( T \) admits a unique invertible outer spectral factor \( \Theta \). Therefore \( \Lambda = W^*T_\Theta^*T\Theta W \). Clearly, there is a one to one correspondence between the set of all solutions to our Nevanlinna-Pick expansion problem and the set of all invertible outer functions \( \Theta \) satisfying \( \Lambda = W^*T_\Theta^*T\Theta W \). Motivated by this we say that \( \Theta \) is a spectral interpolant for the data \( \{A, C, \Lambda\} \) if \( \Theta \) is an invertible outer function in \( H^\infty(U,U) \) satisfying \( \Lambda = W^*T_\Theta^*T\Theta W \).

2 An optimization problem

The following optimization problem plays a fundamental role in our approach to the Nevanlinna-Pick expansion problem
\[
\nu(y, \alpha) = \inf \{(\Lambda x, x) : C(I - \alpha A)^{-1}x = y \}.
\] (2.7)
Here \( \alpha \) is a fixed scalar in \( D \), and \( y \) is a fixed vector in \( U \). This is a standard least squares optimization problem whose solution is given by
\[
x_{opt} = \Lambda^{-1}(I - \bar{\alpha}A^*)^{-1}C^*\Delta y \quad \text{and} \quad \nu(y, \alpha) = (\Delta y, y)
\]
\[
\Delta = (C(I - \alpha A)^{-1}\Lambda^{-1}(I - \bar{\alpha}A^*)^{-1}C^*)^{-1}.
\] (2.8)
To develop a connection to the Nevanlinna-Pick expansion problem, assume that \( \Theta \) is a spectral interpolant for the data \( \{A, C, \Lambda\} \), that is, \( \Theta \) is an invertible outer function satisfying \( \Lambda = W^*T_\Theta^*T\Theta W \). Then \( \Theta(\alpha) \) satisfies the following inequality
\[
\Delta \geq d_\alpha^2\Theta(\alpha)^*\Theta(\alpha)
\] (2.9)
where \( d_\alpha = (1 - |\alpha|^2)^{1/2} \). The previous analysis yields the following maximal principle: If \( \Theta \) is any spectral interpolant for \( \{A, C, \Lambda\} \), then \( \Delta \geq d_\alpha^2\Theta(\alpha)^*\Theta(\alpha) \). We say that \( \Theta \) is a \( \alpha \)-maximal spectral interpolant for \( \{A, C, \Lambda\} \) if \( \Theta \) is a spectral interpolant satisfying \( \Delta = d_\alpha^2\Theta(\alpha)^*\Theta(\alpha) \). The following result shows that the maximal interpolant is unique, and provides a state space realization.
THEOREM 2.1 Let \( \{A, C, \Lambda\} \) be the data set for a Nevanlinna-Pick expansion problem and \( \alpha \in \mathbb{D} \). Moreover, let \( \Omega \) be the function in \( H^\infty(U, U) \) defined by

\[
\Omega(\lambda) = d_\alpha^{-1}(1 - \overline{\alpha} \lambda)C(I - \lambda A)^{-1}\Lambda^{-1}(I - \overline{\alpha} A^*)^{-1}C^*\Delta^{1/2}.
\] (2.10)

Then the following holds.

(i) The inverse \( \Theta(\lambda) = \Omega(\lambda)^{-1} \) is the unique \( \alpha \)-maximal spectral factor for \( \{A, C, \Lambda\} \). In particular, \( \Omega \) is an invertible outer function.

(ii) A realization for \( \Theta \) is given by

\[
\Theta(\lambda) = D - \lambda DC(I - \lambda J)^{-1}(A - \overline{\alpha} I)B(CB)^{-1}
\]

\[
B = \Lambda^{-1}(I - \overline{\alpha} A^*)^{-1}C^*
\]

\[
J = A - (A - \overline{\alpha} I)B(CB)^{-1}C
\]

\[
D = d_\alpha \Delta^{-1/2}(CB)^{-1}.
\] (2.11)

Finally, the operator \( J \) is stable, that is, \( r_{\text{spec}}(J) < 1 \).

If \( \Lambda \) is the Toeplitz matrix given in (1.5) and \( \alpha = 0 \), then \( \Theta \) in (2.11) is the outer spectral factor computed in Levinson filtering.

In general a positive Toeplitz matrix \( T \) may not admit a spectral factorization of the form \( T = T^*_\Phi T \Phi \) where \( \Phi \) is an outer function. However, any positive Toeplitz matrix \( T \) admits a unique maximal outer spectral factor \( \Phi \), that is, \( T \geq T^*_\Phi T \Phi \) and if \( \Psi \) is any other outer spectral factor satisfying \( T \geq T^*_\Psi T \Psi \) with equality if and only if \( \Phi = \Psi \); see [4, 10]. To complete this talk we will show that the \( \alpha \)-maximal outer interpolants can be used to approximate the actual maximal outer spectral factor for \( T \). To be mathematically precise, let \( \{C_k, A_k \text{ on } X_k\}_{k=1}^\infty \) be a sequence of stable, observable realizations and \( W_k \) the observability operator mapping \( X_k \) into \( \ell_2^+_+(U) \) defined in (1.2) by replacing \( \{C, A\} \) with \( \{C_k, A_k\} \). Moreover, let us assume that \( \text{ran} W_k \subset \text{ran} W_{k+1} \) and the spectrum of \( \{A_k\}_{k=1}^\infty \) is contained in some compact set of the open unit disc. Let \( \Phi \) be the maximal outer spectral factor for a positive Toeplitz matrix \( T \). Set \( \Lambda_k = W_k^*TW_k \) and fix \( \alpha \in \mathbb{D} \). Let \( \Theta_k \) be the \( \alpha \)-maximal spectral interpolant for the data \( \{A_k, C_k, \Lambda_k\} \). Then we will show that \( \Theta_k \) converges to \( \Phi \). Moreover, we will show how \( \Theta_k \) can also be used to approximate the point spectrum of \( T \), or equivalently, the eigenvalues of the unitary operator obtained in the Wold decomposition of the Naimark dilation for \( T \). Finally, we will use the cost \( \nu(\alpha) \) for the optimization problem (2.7) with \( |\alpha| = 1 \) to develop an algorithm to compute the point spectrum of \( T \). Some numerical examples will be given.

References


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