Robust Hybrid Control Systems

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“Hybrid” arcs

Continuous arc

Discrete arc

Hybrid arc

Not just for bookkeeping or pedagogy.

Provides a natural way of characterizing convergence of solutions: **graphical convergence** (from set-valued analysis).

Multiple jumps at the same “t” allowed.

P. Collins, MTNS ’04
USCB, NOLCOS ’04
Goebel/T, Automatica ‘06
Looking for hybrid systems for which each sequence of solutions has a subsequence converging to a solution. This property comes “for free” for continuous ODEs and difference equations.

A ball bouncing from a sequence of decreasing heights.
Hybrid arcs generated by data

System data: $\mathcal{H} = (f, g, C, D)$

$\mathcal{H} \begin{cases} 
\dot{x} = f(x) & x \in C \\
x^+ = g(x) & x \in D
\end{cases}$

$x$ may contain variable taking values in a discrete set, timers, etc.

Now looking for conditions on data to guarantee that each sequence of solutions has a subsequence converging to a solution. For simplicity, $f$ and $g$ are single-valued and continuous…

“Lateral” evolution
1) $\dot{x}(t, j) = f(x(t, j)) \quad x(t, j) \in C$
   if $\exists \tau > t : (\tau, j), (t, j) \in \text{dom } x$

“Out of the screen” evolution
2) $x(t, j + 1) = g(x(t, j)) \quad x(t, j) \in D$
   if $(t, j), (t, j + 1) \in \text{dom } x$
Bouncing Ball, First Pass

\[ h = 0, \dot{h} < 0? \]

\[ \ddot{h} = -g \]

\[ \dot{h}^+ = -\gamma \dot{h} \]
\[ \gamma \in (0,1) \]

\[ C_1 := \{(h, \dot{h}) : h > 0\} \]

\[ D_1 := \{(h, \dot{h}) : h = 0, \dot{h} < 0\} \]
Bouncing Ball, Pass one (cont’d)

Ball bouncing from a sequence of decreasing heights:
converges to an instantaneous Zeno solution that remains at zero height.
But this is not a solution of the BB since the origin does not belong to $D$. 
Bouncing Ball, Second Pass

\[ h = 0, \dot{h} \leq 0? \]

\[ \dot{h} = -\gamma \dot{h} \]

\[ \gamma \in (0,1) \]

\[ C_2 := \{(h, \dot{h}) : h \geq 0\} \]

\[ C_2 = \overline{C}_1 \]

\[ D_2 := \{(h, \dot{h}) : h = 0, \dot{h} \leq 0\} \]

\[ D_2 = \overline{D}_1 \]

Admits instantaneous Zeno solution at origin.
Conditions on data for convergence property

System data: $\mathcal{H} = (f, g, C, D)$

If $C$ and $D$ are closed then each sequence of solutions has a subsequence converging to a solution. Otherwise, no guarantee.

This leads to a natural notion of “generalized solutions” (a la Filippov or Krasovskii for discontinuous ODEs) for hybrid systems:

Hybrid “Krasovskii” solutions satisfy:

$$
\mathcal{H} \quad \left\{ \begin{array}{l}
\dot{x} = f(x) \quad x \in \overline{C} \\
\dot{x}^+ = g(x) \quad x \in \overline{D}.
\end{array} \right.
$$

Natural from a control point of view since they agree with “hybrid Hermes” solutions, i.e., the zero noise graphical limit of solutions to

$$
\mathcal{H}_e \quad \left\{ \begin{array}{l}
\dot{x} = f(x + e) \quad x + e \in C \\
\dot{x}^+ = g(x + e) \quad x + e \in D.
\end{array} \right.
$$
On “generalized” solutions

Biases against (stemming from ODEs):

1) “I don’t work with discontinuous continuous-time systems, so I have no use for generalized solutions.”

2) “The solution notion is flawed because it is too restrictive; e.g., it prevents point stabilization by state feedback for mobile robots.”

Rejoinders:

1) For hybrid systems, generalized solutions are relevant even for systems without discontinuities.

2) Every asymptotically controllable nonlinear system (e.g., the mobile robot model) can be stabilized using hybrid feedback with closed flow and jump sets, i.e., using generalized solutions.

3) Value added: Converse Lyapunov theorems, LaSalle’s invariance principle, generic robustness of asymptotic stability become free.
Generalized solutions matter, Example 1

\[ f(x) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x \]
\[ g(x) = \begin{bmatrix} 0 & 0 \\ -\varepsilon & 0 \end{bmatrix} x, \quad 0 < \varepsilon \ll 1 \]

\[ C := \mathbb{R}^2 \setminus D \]
\[ D := \{(x_1, x_2) : x_2 = 0\}. \]

Generalized solutions: \((\overline{D} = D, \quad \overline{C} = \mathbb{R}^2)\)

original solutions plus:
and combinations thereof…

“Objection! I would implement with ‘zero crossing detection’ (in Simulink.)”

Reply: Put it in your model, e.g.
\[
\begin{align*}
\dot{x}_3 &= 0 \\
x_3^+ &= \text{sgn}(-\varepsilon_1)
\end{align*}
\]
\[ C := \{x : x_2x_3 \geq 0\} \quad \text{Or: make } D \text{ “thick perhaps a sector.} \]

Benefits: Explicit robustness, Lyapunov function exists, LaSalle available.
Generalized solutions matter, Example 2

Target acquisition with obstacle avoidance

Mode 1: drive to target

Mode 2: drive away from obstacle

\[ Q := \{1, 2\}, \quad x := \begin{bmatrix} z \\ q \end{bmatrix} \in \mathbb{R}^2 \times Q \]

\[ \begin{bmatrix} z^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} z \\ 2 \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix} \in D_1 \times \{1\} \]

\[ \begin{bmatrix} z^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} z \\ 1 \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix} \in D_2 \times \{2\} \]

\[ \dot{z} = f_q(z), \quad \dot{q} = 0 \]

on \( (\mathbb{R}^2 \setminus D_1) \times \{1\} \cup (\mathbb{R}^2 \setminus D_2) \times \{2\} \)

\[ \vec{C} = \mathbb{R}^2 \times Q. \]
Extra generalized solutions

\[
\overline{C} = R^2 \times Q
\]

Generalized solutions:

Simple way to get rid of extra solutions:
(preferred to zero cross detection or thickening here)

Benifits as before:
Lyapunov, LaSalle,…

\[
\begin{align*}
D_{1,new} & := D_1 + \quad \text{inside of circle it describes.} \\
D_{2,new} & := D_2 + \quad \text{outside of circle it describes.} \\
C_1 & := \overline{R^2 \setminus D_{1,new}} \\
C_2 & := \overline{R^2 \setminus D_{2,new}}, \quad \text{ } C = (C_1 \times \{1\}) \cup (C_2 \times \{2\})
\end{align*}
\]
LaSalle’s invariance principle

**Theorem:** If
\[
\left< \nabla V(x), f(x) \right> \leq u_c(x) \leq 0 \quad x \in \overline{C}
\]
\[
V(g(x)) - V(x) \leq u_d(x) \leq 0 \quad x \in \overline{D}
\]

then each complete, bounded trajectory converges to the largest weakly invariant set contained in
\[
M_r := V^{-1}(r) \cap u_c^{-1}(0) \cup [u_d^{-1}(0) \cap g(u_d^{-1}(0))]
\]
for some \( r \)

**Example:**
\[
f(x) = g(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x, \quad V(x) = x^T x
\]

(solutions are not unique)

Largest weakly invariant set contained in \( M_r \) excludes all points in green except the one on the vertical axis.
A Converse Lyapunov theorem

Theorem: If a compact set is asymptotically stable (locally stable + attractive) (using generalized solutions) then there exists a smooth Lyapunov function.

Let \( \omega : O \rightarrow R_{\geq 0} \) be

- positive definite with respect to the compact set,
- proper w.r.t. \( O \), which is an open set related to the basin of attraction, the latter which is open relative to \( \overline{C} \cup \overline{D} \).

There exists a smooth function \( V : O \rightarrow R_{\geq 0} \) & \( \alpha_1, \alpha_2 \in K_\infty : \)

1) \( \alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x)) \quad x \in O \)
2) \( \left\langle \nabla V(x), f(x) \right\rangle \leq -V(x) \quad x \in \overline{C} \)
3) \( V(g(x)) \leq \exp(-1)V(x) \quad x \in \overline{D} \)

Significance: Can be used to show a wide variety of robustness properties.

Note: \( \omega(x(t, j)) \leq \alpha_1^{-1}(\exp(-t - j)\alpha_2(\omega(x(0,0)))) \)
Autonomous vs. exogenous perturbations

Lyapunov functions can be used to show robustness to ‘autonomous’ perturbations, i.e., inflations of \((f,g,C,D)\):

Local separation principles attainable.

Solutions with exogenous disturbances covered by those of inflated systems:

\[
\begin{align*}
    F_\delta(x) &:= \text{co } f(x + \delta B) + \delta B \\
    G_\delta(x) &:= g(x + \delta B) + \delta B \\
    C_\delta &:= \{ x : (x + \delta B) \cap C \neq \emptyset \} \\
    D_\delta &:= \{ x : (x + \delta B) \cap D \neq \emptyset \}
\end{align*}
\]

\[
\begin{align*}
    \dot{x} &= f(x + e) \quad x + e \in C \\
    x^+ &= g(x + e) \quad x + e \in D.
\end{align*}
\]

However, existence is not guaranteed for arbitrary signal \(e\):

Message: make flow and jump sets overlap, or keep measurements out of the switching conditions, via sample and hold or filtering measurements.
Thermostat control and 2-measure “stability”

The set of plant temperatures in the range [20,25] is not forward invariant, and thus not asymptotically stable. Nevertheless:

$$|\xi(t, j)|_{[20,25]} \leq \beta(\omega_2(x(0,0)), t, j)$$

must be positive outside of dashed blue $$\beta \in KLL$$

A smooth Lyapunov function ensues:

1) $$\alpha_1(|\xi|_{[20,25]}) \leq V(x) \leq \alpha_2(\omega_2(x))$$
2) $$\langle \nabla V(x), f(x) \rangle \leq -V(x) \quad x \in C$$
3) $$V(g(x)) \leq \exp(-1)V(x) \quad x \in D$$

and gives several robustness consequences:

- singular perturbations (actuators or sensors),
- slowly-varying parameters, sample & hold,
- small delays, temporal regularization, etc.
“Hybrid handoff” from global to local controller

Full state feedback:

\[ D_{\text{global}} = R^n \setminus C_{\text{global}} \]

\[ C_{\text{local}} = R^n \setminus D_{\text{local}} \]

Trajectory due to local controller

Extension to output feedback in the presence of norm observers.

C. Prieur/L. Praly ‘99

C. Prieur/T, in preparation
Smooth “patchy” control Lyapunov functions

Theorem: Every asymptotically controllable (to a compact set) nonlinear system admits a smooth “patchy control Lyapunov function” and, in turn, is robustly stabilizable by hybrid feedback.

Remark: In a sense, the latter was already known, from results by Sontag and also Clarke et al., on the robustness that results from implementing certain discontinuous controllers with sample and hold: “Sample fast, but not too fast”. Indeed, sample and hold produces a hybrid system:

\[
\begin{align*}
\dot{x} &= f(x, u) \quad \tau \in [0, T] \\
\dot{\tau} &= 1 \\
u^+ &= g(x) \quad \tau \in [T, \infty) \\
\tau^+ &= 0
\end{align*}
\]

Using “patches” instead of sample and hold allows you to keep fast sampling without sacrificing robustness with respect to measurement noise.
Smooth “patchy” control Lyapunov functions

**Ingredients:**
- An ordered index set $Q$
- (L.f.) Families of open patches that cover: $\overline{\Omega}'_q \subset \Omega'_q \leftarrow$ smooth boundary
- Smooth functions $V_q$ defined on $\Omega'_q$

**Mechanism:**
- Start in $\Omega'_q$ stay strictly in $\Omega'_q$ until move to new patch.
- Switch to index $r$ when reach $\Omega'_r$

1) $\alpha_1(\|x\|) \leq V_q(x) \leq \alpha_2(\|x\|) \quad x \in \Omega'_q \setminus \bigcup_{r>q} \Omega'_r$

2a) $\left\langle \nabla V_q(x), f(x,u_{q,x}) \right\rangle \leq -\rho(\|x\|) \quad x \in \Omega'_q \setminus \bigcup_{r>q} \Omega'_r$

2b) $\left\langle n_q(x), f(x,u_{q,x}) \right\rangle \leq -\rho(\|x\|) \quad x \in \partial \Omega'_q \setminus \bigcup_{r>q} \Omega'_r$

Nonholonomic integrator: 3 patches, Artstein’s circles: 2 patches.
Patches without control Lyapunov functions

Any mechanism guaranteeing flow from a patch to a “higher” patch is fine. Inverted pendulum on a cart: 3 patches. Robustness to singular perturbation to smoothen control signal.
Robust decision making in noisy environments

depends on $q$
Conclusions

- We have emphasized robustness in hybrid systems, especially hybrid control systems.

- Robustness in hybrid systems relies on closed flow and jump sets, or “generalized solutions”.

- Robustness provides many of the stability analysis tools our community has exploited for continuous-time control systems.