The Behavioral Approach to Systems Theory

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Lecture 4: Bilinear and quadratic differential forms

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Part I: Basics
Outline

Motivation and aim

Definition

Two-variable polynomial matrices

The calculus of B/QDFs
Dynamics and functionals in systems and control

**Instances:** Lyapunov theory, performance criteria, etc.

Linear case $\implies$ *quadratic* and *bilinear* functionals.
Dynamics and functionals in systems and control

**Instances:** Lyapunov theory, performance criteria, etc.

Linear case $\Rightarrow$ *quadratic* and *bilinear* functionals.

**Usually:** state-space equations, constant functionals.

However, tearing and zooming $\Rightarrow$ state space eq.s
Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\implies$ quadratic and bilinear functionals.

Usually: state-space equations, constant functionals.

However, tearing and zooming $\implies$ state space eq.s

¡High-order differential equations!

...involving also latent variables...
Example: a mechanical system

\[ m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_1 w_2 = 0 \]

\[ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 = 0 \]
Example: a mechanical system

\[ m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_1 w_2 = 0 \]

\[ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 = 0 \]

\[ m_1 m_2 \frac{d^4 w}{dt^4} + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2 w}{dt^2} + k_1 k_2 w = 0 \]
Example: a mechanical system

\[ m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_1 w_2 = 0 \]
\[ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 = 0 \]

\[ m_1 m_2 \frac{d^4 w}{dt^4} + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2 w}{dt^2} + k_1 k_2 w = 0 \]

¿Stability, stored energy, conservation laws?
Aim

An effective algebraic representation of bilinear and quadratic functionals of the system variables and their derivatives:

Operations/properties of functionals
\[\supseteq\]
algebraic operations/properties of representation

...a calculus of these functionals!
Outline

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The calculus of B/QDFs
Bilinear differential forms (BDFs)

\[ \Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \right\}_{k,\ell=0,\ldots,L} \]

\[ L_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \]

\[ L_\Phi(w_1, w_2) := \left[ w_1^\top \quad \frac{dw_1}{dt}^\top \quad \ldots \right] \left[ \begin{array}{ccc}
\Phi_{0,0} & \Phi_{0,1} & \ldots \\
\Phi_{1,0} & \Phi_{1,1} & \ldots \\
\vdots & \vdots & \ddots \\
\Phi_{k,0} & \Phi_{k,1} & \ldots \\
\vdots & \vdots & \ddots 
\end{array} \right] \left[ \begin{array}{c}
W_2 \\
\frac{dw_2}{dt} \\
\vdots 
\end{array} \right] \]

\[ = \sum_{k,\ell} \left( \frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left( \frac{d^\ell}{dt^\ell} w_2 \right) \]
Quadratic differential forms (QDFs)

\[ \Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w \times w} \right\}_{k,\ell=0,...,L} \text{ symmetric, i.e. } \Phi_{k,\ell} = \Phi_{\ell,k}^\top \]

\[ Q_{\Phi} : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \]

\[ Q_{\Phi}(w) := \left[w^\top \frac{dw}{dt}^\top \ldots \right] \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \ldots \\ \Phi_{1,0} & \Phi_{1,1} & \ldots \\ \vdots & \vdots & \ldots \\ \Phi_{k,0} & \Phi_{k,1} & \ldots \\ \vdots & \vdots & \ldots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix} \]

\[ = \sum_{k,\ell=0}^{L} \left( \frac{d^k}{dt^k} W \right)^\top \Phi_{k,\ell} \left( \frac{d^\ell}{dt^\ell} W \right) \]
Example: total energy in mechanical system

\[
\frac{1}{2} \left[ \left( \frac{d}{dt} w_1 \right)^2 + \left( \frac{d}{dt} w_2 \right)^2 \right] + \frac{1}{2} \left[ k_1 w_1^2 + k_2 w_2^2 \right]
\]

\[
\begin{bmatrix}
w_1 & w_2 & \frac{d}{dt} w_1 & \frac{d}{dt} w_2
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} k_1 & 0 & 0 & 0 \\
0 & \frac{1}{2} k_2 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
\frac{d}{dt} w_1 \\
\frac{d}{dt} w_2
\end{bmatrix}
\]
Outline

Motivation and aim

Definition

Two-variable polynomial matrices

The calculus of B/QDFs
Two-variable polynomial matrices for BDFs

\( \{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \}_{k,\ell=0,\ldots,L} \)

\[
L_{\Phi}(w_1, w_2) = \sum_{k,\ell=0}^{L} \left( \frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w_2
\]

\[
\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell
\]
Two-variable polynomial matrices for BDFs

\[
\left\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \right\}_{k,\ell=0,\ldots,L}
\]

\[
L_{\Phi}(w_1, w_2) = \sum_{k,\ell=0}^{L} \left( \frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w_2
\]

\[
\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell
\]
Two-variable polynomial matrices for BDFs

\[
\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \}_{k,\ell=0,\ldots,L}
\]

\[
L_{\Phi}(w_1, w_2) = \sum_{k,\ell=0}^{L} \left( \frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w_2
\]

\[
\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell
\]

2-variable polynomial matrix associated with \( L_{\Phi} \)
Two-variable polynomial matrices for QDFs

\[ \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w \times w} \right\}_{k,\ell=0,...,L} \text{ symmetric (} \Phi_{k,\ell} = \Phi_{\ell,k}^\top \) \]

\[ Q_{\phi}(w) = \sum_{k,\ell=0}^{L} \left( \frac{d^k}{dt^k} w \right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w \]

\[ \Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell \]

**symmetric:** \[ \Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top \]
Example: total energy in mechanical system

\[ Q_E(w_1, w_2) = \begin{bmatrix} w_1 & w_2 & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \end{bmatrix} \]

\[ E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2} k_1 & 0 \\ 0 & \frac{1}{2} k_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \zeta \eta & 0 \\ 0 & \frac{1}{2} \zeta \eta \end{bmatrix} \]
Historical intermezzo

stability tests ('60s)

path integrals ('60s)

Lyapunov functionals ('80s)

QDFs (1998)
Historical intermezzo

stability tests (‘60s)

path integrals (‘60s)

Lyapunov functionals (‘80s)

QDFs (1998)
Historical intermezzo

- path integrals (‘60s)
- stability tests (‘60s)
- Lyapunov functionals (‘80s)
- QDFs (1998)
Historical intermezzo

Lyapunov functionals (‘80s)

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Historical intermezzo

- Lyapunov functionals ('80s)
- Path integrals ('60s)
- Stability tests ('60s)
- QDFs (1998)
Outline

Motivation and aim

Definition

Two-variable polynomial matrices

The calculus of B/QDFs
The calculus of B/QDFs

Using powers of \( \zeta \) and \( \eta \) as placeholders,

\[ \text{B/QDF} \leftrightarrow \text{two-variable polynomial matrix} \]
The calculus of B/QDFs

Using powers of $\zeta$ and $\eta$ as placeholders,

$B/QDF \leftrightarrow \text{two-variable polynomial matrix}$

Operations and properties of B/QDF $\leftrightarrow$ algebraic operations/properties on two-variable matrix
\[ \Phi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]. \text{ \textbullet \: \Phi \text{ derivative of } Q_{\Phi}:} \]

\[ Q_{\Phi} : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \]

\[ Q_{\Phi}(w) := \frac{d}{dt}(Q_{\Phi}(w)) \]

\[ \Phi(\zeta, \eta) = (\zeta + \eta)\Phi(\zeta, \eta) \]

Two-variable version of Leibniz’s rule
\[ \mathcal{D}(\mathbb{R}, \mathbb{R}^\bullet) \mathcal{C}^\infty\text{-compact-support trajectories} \]

\[ L_\Phi : \mathcal{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathcal{D}(\mathbb{R}, \mathbb{R}^{w_2}) \rightarrow \mathcal{D}(\mathbb{R}, \mathbb{R}) \]

\[ \int L_\Phi : \mathcal{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathcal{D}(\mathbb{R}, \mathbb{R}^{w_2}) \rightarrow \mathbb{R} \]

\[ \int L_\Phi(w_1, w_2) := \int_{-\infty}^{+\infty} L_\Phi(w_1, w_2) \, dt \]

Analogous for QDFs
Part II: Applications
Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction
Nonnegativity and positivity along a behavior

\[ Q_\Phi \geq 0 \text{ if } Q_\Phi(w) \geq 0 \forall w \in \mathcal{B} \]
Nonnegativity and positivity along a behavior

\[ Q_\Phi \geq 0 \text{ if } Q_\Phi(w) \geq 0 \quad \forall \ w \in \mathcal{B} \]

\[ Q_\Phi > 0 \text{ if } Q_\Phi \geq 0, \text{ and } [Q_\Phi(w) = 0] \implies [w = 0] \]
Nonnegativity and positivity along a behavior

\[ Q_\Phi \geq 0 \text{ if } Q_\Phi(w) \geq 0 \forall w \in \mathcal{B} \]

\[ Q_\Phi > 0 \text{ if } Q_\Phi \geq 0, \text{ and } [Q_\Phi(w) = 0] \implies [w = 0] \]

**Prop.:** Let \( \mathcal{B} = \ker R(\frac{d}{dt}) \). Then \( Q_\Phi \geq 0 \) iff there exist \( D \in \mathbb{R}^{\mathbb{w} \times \mathbb{w}}[\xi] \), \( X \in \mathbb{R}^{\mathbb{w} \times \mathbb{w}}[\zeta, \eta] \) such that

\[
\Phi(\xi, \eta) = D(\xi)^\top D(\eta) + R(\zeta)^\top X(\zeta, \eta) + X(\eta, \zeta)^\top R(\eta)
\begin{align*}
\geq 0 & \text{ for all } w \\
= 0 & \text{ if evaluated on } \mathcal{B}
\end{align*}
\]
Lyapunov theory

\( \mathcal{B} \) autonomous is \textit{asymptotically stable} if
\[
\lim_{t \to \infty} w(t) = 0 \quad \forall \ w \in \mathcal{B}
\]

\( \mathcal{B} = \ker R(\frac{d}{dt}) \) stable \iff \det(R) \text{ Hurwitz}
Lyapunov theory

\[ \mathcal{B} \text{ autonomous is asymptotically stable if } \lim_{t \to \infty} w(t) = 0 \quad \forall \ w \in \mathcal{B} \]

\[ \mathcal{B} = \ker R \left( \frac{d}{dt} \right) \text{ stable } \iff \det(R) \text{ Hurwitz} \]

**Theorem:** \( \mathcal{B} \) asymptotically stable iff exists \( Q_\Phi \) such that \( Q_\Phi \geq 0 \) and \( Q_\Phi < 0 \)
Example

\[ \mathcal{B} = \ker \left( \frac{d^2}{dt^2} + 3 \frac{d}{dt} + 2 \right) \]

\[ r(\xi) = \xi^2 + 3\xi + 2 \]
Example

\[ \mathcal{B} = \ker \left( \frac{d^2}{dt^2} + 3 \frac{d}{dt} + 2 \right) \]

\[ r(\xi) = \xi^2 + 3\xi + 2 \]

Choose \( \Psi(\zeta, \eta) \) s.t. \( Q_{\mathcal{B}} \subset 0 \), e.g. \( \Psi(\zeta, \eta) = -\zeta \eta \);
Choose $\Psi(\zeta, \eta)$ s.t. $Q_{\Psi} \Brian{\B} < 0$, e.g. $\Psi(\zeta, \eta) = -\zeta \eta$;

Find $\Phi(\zeta, \eta)$ s.t. $\frac{d}{dt} Q_{\Phi}(w) = Q_{\Psi}(w)$ for all $w \in \B$:

$$(\zeta + \eta) \Phi(\zeta, \eta) = \Psi(\zeta, \eta) + r(\zeta)x(\eta) + x(\zeta)r(\eta)$$

$= 0$ on $\B$
Example

\[ \mathcal{B} = \ker \left( \frac{d^2}{dt^2} + 3 \frac{d}{dt} + 2 \right) \]

\[ r(\xi) = \xi^2 + 3\xi + 2 \]

Choose \( \Psi(\zeta, \eta) \) s.t. \( Q_\Psi^\mathcal{B} < 0 \), e.g. \( \Psi(\zeta, \eta) = -\zeta \eta \);

Find \( \Phi(\zeta, \eta) \) s.t. \( \frac{d}{dt} Q_\Phi(w) = Q_\Psi(w) \) for all \( w \in \mathcal{B} \):

\[ (\zeta + \eta) \Phi(\zeta, \eta) = \Psi(\zeta, \eta) + r(\zeta) x(\eta) + x(\zeta) r(\eta) = 0 \] on \( \mathcal{B} \)

\[ \frac{d}{dt} Q_\Phi(w) = Q_\Psi(w) \] for all \( w \in \mathcal{B} \)
Example

\[ \mathcal{B} = \ker \left( \frac{d^2}{dt^2} + 3 \frac{d}{dt} + 2 \right) \quad r(\xi) = \xi^2 + 3\xi + 2 \]

Choose \( \Psi(\zeta, \eta) \) s.t. \( \overline{\mathcal{B}} \frac{\mathcal{B}}{Q_{\Psi}} < 0 \), e.g. \( \Psi(\zeta, \eta) = -\zeta \eta \);

Find \( \Phi(\zeta, \eta) \) s.t. \( \frac{d}{dt} Q_{\Phi}(w) = Q_{\Psi}(w) \) for all \( w \in \mathcal{B} \):

\[
(\zeta + \eta) \Phi(\zeta, \eta) = \Psi(\zeta, \eta) + r(\zeta) x(\eta) + x(\zeta) r(\eta)
\]

\( = 0 \) on \( \mathcal{B} \)

Equivalent to solving polynomial Lyapunov equation

\[
0 = \Psi(-\xi, \xi) + r(-\xi) x(\xi) + x(-\xi) r(\xi)
\]

\[
\xi^2 \quad \xi^2 - 3\xi + 2 \quad \xi^2 + 3\xi + 2
\]

\( \sim x(\xi) = \frac{1}{6} \xi \)
Choose $\Psi(\zeta, \eta)$ s.t. $Q_\Psi < 0$, e.g. $\Psi(\zeta, \eta) = -\zeta \eta$;

Find $\Phi(\zeta, \eta)$ s.t. $\frac{d}{dt} Q_\Phi(w) = Q_\Psi(w)$ for all $w \in \mathcal{B}$:

$$(\zeta + \eta) \Phi(\zeta, \eta) = \Psi(\zeta, \eta) + r(\zeta)x(\eta) + x(\zeta) r(\eta) = 0 \text{ on } \mathcal{B}$$

$$\Phi(\zeta, \eta) = \frac{-\zeta \eta + (\zeta^2 + 3\zeta + 2)\frac{1}{6} \eta + \frac{1}{6} \zeta(\eta^2 + 3\eta + 2)}{\zeta + \eta}$$

$$= \frac{1}{6} \zeta \eta + \frac{1}{3} > 0$$
State-space case

\[
\left( \frac{d}{dt} I_x - A \right) x = 0 \quad \sim \quad R(\xi) = \xi I_x - A
\]

- Choose \( Q < 0 \);
- Solve polynomial Lyapunov equation

\[
(\xi I_x - A)^\top P + P(\xi I_x - A) = -A^\top P - PA = Q
\]

equivalent with matrix Lyapunov equation!
- Lyapunov functional is

\[
x^\top (-P) x
\]
Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction
Dissipativity theory

**System**

Power is **supplied**

\[ \text{energy is stored} \]

**RLC circuits**  
**Power**  
\[ V^\top I \]

**Storage in capacitors and inductors**

**Mechanical system**  
**Power**  
\[ F^\top v + (\frac{d}{dt} \phi)^\top T \]

**Potential + kinetic**
Setting the stage

Controllable system

\[ w = M \left( \frac{d}{dt} \right) \ell \sim M(\xi) \]

Power (‘supply rate’)

\[ Q_\Phi (w) \sim \Phi(\zeta, \eta) \]
Setting the stage

Controllable system

\[ w = M\left(\frac{d}{dt}\ell\right) \sim M(\xi) \]

Power ('supply rate')

\[ Q_\Phi(w) \sim \Phi(\zeta, \eta) \]

\[
Q_\Phi(w) = Q_\Phi(M\left(\frac{d}{dt}\ell\right))
\]

\[
\Phi'(\zeta, \eta) := M(\zeta)^\top \Phi(\zeta, \eta)M(\eta)
\]

\( Q_\Phi \) acts on free variable \( \ell \), i.e. \( C^\infty \)
Setting the stage

Controllable system

\[ w = M\left( \frac{d}{dt} \right) \ell \sim M(\xi) \]

Power (‘supply rate’)

\[ Q_\Phi(w) \sim \Phi(\zeta, \eta) \]

\[ Q_\Phi(w) = Q_\Phi(M(\frac{d}{dt})\ell) \]

\[ \Phi'(\zeta, \eta) := M(\zeta)^\top \Phi(\zeta, \eta)M(\eta) \]

\( Q_\Phi \) acts on free variable \( \ell \), i.e. \( C^\infty \)
Dissipation inequality

\[ \dot{Q} \Psi \leq Q \Phi \]
Rate of storage increase \[ \leq \] supply

\[ Q \Delta \geq 0 \] and
\[ \int Q \Delta \, dt = \int Q \Phi \, dt \]
$Q_\psi$ is **storage function** for the supply $Q_\Phi$ if

$$\frac{d}{dt} Q_\psi \leq Q_\Phi$$

Rate of storage increase $\leq$ supply
\( Q_\Psi \) is storage function for the supply \( Q_\Phi \) if

\[
\frac{d}{dt} Q_\Psi \leq Q_\Phi
\]

Rate of storage increase \( \leq \) supply

\( Q_\Delta \) is dissipation function for \( Q_\Phi \) if

\[
Q_\Delta \geq 0 \quad \text{and} \quad \int Q_\Delta \, dt = \int Q_\Phi \, dt
\]
Characterizations of dissipativity

**Theorem:** The following conditions are equivalent:

- \( \int_{-\infty}^{+\infty} Q_\Phi(\ell) \, dt \geq 0 \) for all \( C^\infty \) compact-support \( \ell \);
- \( Q_\Phi \) admits a storage function;
- \( Q_\Phi \) admits a dissipation function.

Also, storage and dissipation functions are one-one:

\[
\frac{d}{dt} Q_\Psi = Q_\Phi - Q_\Delta \\
(\zeta + \eta) \psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta)
\]
Example: mechanical systems

\[ M \frac{d^2}{dt^2} q + D \frac{d}{dt} q + Kq = F \]

\[
\begin{bmatrix}
F \\
q
\end{bmatrix} = 
\begin{bmatrix}
M \frac{d^2}{dt^2} + D \frac{d}{dt} + K
\end{bmatrix} \ell
\]

Storage function

\[ \Phi(\zeta, \eta) = \frac{1}{2} \left( M \zeta^2 + D \zeta + K \right) \eta^T + \frac{1}{2} \zeta \left( M \eta^2 + D \eta + K \right) \]

\[ \Delta(\zeta, \eta) = \frac{1}{2} \left( D^\top + D \right) \zeta \eta \]

Total energy

\[ \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta) \]

\[ \zeta + \eta = \frac{1}{2} M \zeta \eta + \frac{1}{2} K \]
Example: mechanical systems

\[ M \frac{d^2 q}{dt^2} + D \frac{d}{dt} q + K q = F \]

\[ \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M \frac{d^2}{dt^2} + D \frac{d}{dt} + K \end{bmatrix} \ell \]

Supply rate: power

\[ F^\top \left( \frac{d}{dt} q \right) = \left( M \frac{d^2}{dt^2} \ell + D \frac{d}{dt} \ell + K \ell \right)^\top \left( \frac{d}{dt} \ell \right) \]

corresponding to

\[ \Phi(\zeta, \eta) = \frac{1}{2} (M \zeta^2 + D \zeta + K)^\top \eta + \frac{1}{2} \zeta (M \eta^2 + D \eta + K) \]
Example: mechanical systems

\[ M \frac{d^2}{dt^2} q + D \frac{d}{dt} q + Kq = F \]

\[
\begin{bmatrix}
F \\
q
\end{bmatrix} = \begin{bmatrix}
M \frac{d^2}{dt^2} + D \frac{d}{dt} + K
\end{bmatrix} \ell
\]

\[
\Phi(\zeta, \eta) = \frac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2}\zeta(M\eta^2 + D\eta + K)
\]
Example: mechanical systems

\[ M \frac{d^2}{dt^2} q + D \frac{d}{dt} q + K q = F \]

\[ \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M \frac{d^2}{dt^2} + D \frac{d}{dt} + K \end{bmatrix} \ell \]

\[ \Phi(\zeta, \eta) = \frac{1}{2} (M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2} \zeta (M\eta^2 + D\eta + K) \]

If dissipation inequality

\[ \Phi(\zeta, \eta) = (\zeta + \eta) \psi(\zeta, \eta) + \Delta(\zeta, \eta) \]

holds, then

\[ \Phi(-\xi, \xi) = -\frac{1}{2} \xi^2 (D^\top + D) = \Delta(-\xi, \xi) \]

\[ \implies \Delta(\zeta, \eta) = \frac{1}{2} (D^\top + D) \zeta \eta \]

**Spectral factorization** of \( \Phi(-\xi, \xi) \) is key
Example: mechanical systems

\[ M \frac{d^2}{dt^2} q + D \frac{d}{dt} q + K q = F \]

\[
\begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M \frac{d^2}{dt^2} + D \frac{d}{dt} + K \end{bmatrix} \ell
\]

\[
\Phi(\zeta, \eta) = \frac{1}{2} (M\zeta^2 + D\zeta + K) \top \eta + \frac{1}{2} \zeta (M\eta^2 + D\eta + K)
\]

\[
\Delta(\zeta, \eta) = \frac{1}{2} (D \top + D) \zeta \eta
\]
Example: mechanical systems

\[
M \frac{d^2}{dt^2} q + D \frac{d}{dt} q + Kq = F \\
\begin{bmatrix}
F \\
q
\end{bmatrix} = \begin{bmatrix}
M \frac{d^2}{dt^2} + D \frac{d}{dt} + K
\end{bmatrix} \ell
\]

\[
\Phi(\zeta, \eta) = \frac{1}{2} (M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2} \zeta(M\eta^2 + D\eta + K)
\]

\[
\Delta(\zeta, \eta) = \frac{1}{2} (D^\top + D) \zeta \eta
\]

Storage function

\[
\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - \Delta(\zeta, \eta)}{\zeta + \eta} = \frac{1}{2} M\zeta \eta + \frac{1}{2} K
\]

Total energy
Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction
Balancing

A minimal and stable realization \((A, B, C, D)\) is \textbf{balanced} if exist \(\sigma_i \in \mathbb{R}\) such that
\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0
\]
and moreover
\[
A\Sigma + \Sigma A^\top + BB^\top = 0
\]
\[
A^\top \Sigma + \Sigma A + C^\top C = 0
\]
where \(\Sigma := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\)
Balancing

A minimal and stable realization \((A, B, C, D)\) is balanced if exist \(\sigma_i \in \mathbb{R}\) such that

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0
\]

and moreover

\[
A\Sigma + \Sigma A^\top + BB^\top = 0 \\
A^\top \Sigma + \Sigma A + C^\top C = 0
\]

where \(\Sigma := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\)

Balancing \(\equiv\) choice of basis of state space diagonalizing the Gramians
Balancing

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\end{align*}
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Balancing \(\equiv\) choice of basis of state space
diagonalizing the Gramians

\(\equiv\) choice of state map!
The controllability Gramian $K$

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell$$

where $GCD(p, q) = 1$, $p$ stable, $\deg(q) \leq \deg(p) =: n$
The controllability Gramian $K$

\[ p(\frac{d}{dt})y = q(\frac{d}{dt})u \]

\[
\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q(\frac{d}{dt}) \\ p(\frac{d}{dt}) \end{bmatrix} \ell
\]

where $GCD(p, q) = 1$, $p$ stable, $\deg(q) \leq \deg(p) =: n$

In state-space framework, $K$ is defined as

\[
\inf_{u} \int_{-\infty}^{0} u(t)^2 dt =: x_0^\top K x_0
\]

where $u$ is such that $x(-\infty) \sim x(0) = x_0$
The controllability Gramian $K$

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u
\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell$$

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In our framework: let $\ell \in C^\infty(\mathbb{R}, \mathbb{R})$. Then $Q_K$ is QDF such that

$$\inf_{\ell'} \int_{-\infty}^{0} \left( p\left(\frac{d}{dt}\right)\ell' \right) dt =: Q_K(\ell)(0)$$

where $\ell' \in C^\infty(\mathbb{R}_+, \mathbb{R})$ is such that $\ell'|_{[0, +\infty)} = \ell|_{[0, +\infty)}$
The controllability Gramian $K$

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u \]

\[ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell \]

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where $\ell' \in C^\infty(\mathbb{R}^+, \mathbb{R})$ is such that $\ell'|_{[0, +\infty)} = \ell|_{[0, +\infty)}$

¿How to compute $K(\zeta, \eta)$?
Computation of $K(\zeta, \eta)$

$$\inf_{\ell'} \int_{-\infty}^{0} \left(p\left(\frac{d}{dt}\ell'\right)\right) dt =: Q_{K}(\ell)(0)$$
Computation of $K(\zeta, \eta)$

$$\inf_{\ell'} \int_{-\infty}^{0} \left( p \left( \frac{d}{dt} \ell' \right) \right) dt =: Q_K(\ell)(0)$$

Since $p(-\xi)p(\xi) = p(\xi)p(-\xi)$, exists $K' \in \mathbb{R}[\zeta, \eta]$ such that

$$p(\zeta)p(\eta) - p(-\zeta)p(-\eta) = (\zeta + \eta)K(\zeta, \eta)$$
Computation of $K(\zeta, \eta)$

$$\inf_{\ell'} \int_{-\infty}^{0} \left( p \left( \frac{d}{dt} \right) \ell' \right) dt =: Q_K(\ell)(0)$$

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$$p(\zeta)p(\eta) - p(-\zeta)p(-\eta) = (\zeta + \eta)K(\zeta, \eta)$$

Consequently,

$$\int_{-\infty}^{0} \left( p \left( \frac{d}{dt} \right) \ell' \right) dt = \int_{-\infty}^{0} \left( p \left( -\frac{d}{dt} \right) \ell' \right) dt + Q_K(\ell')(0)$$

minimized for the $\ell'$ in $\ker p \left( -\frac{d}{dt} \right)$ with the given initial conditions.
Computation of $K(\zeta, \eta)$

$$\inf_{\ell'} \int_{-\infty}^{0} \left( p(\frac{d}{dt})\ell' \right) dt =: Q_K(\ell)(0)$$

Since $p(-\xi)p(\xi) = p(\xi)p(-\xi)$, exists $K' \in \mathbb{R}[\zeta, \eta]$ such that

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Highest power of $\zeta$ and $\eta$ in $K$ is $n - 1$

$\implies Q_K$ is quadratic function of $\frac{d^j \ell}{dt^j}, j = 0, \ldots, n-1$
Computation of $K(\zeta, \eta)$

$$\inf_{\ell'} \int_{-\infty}^{0} \left( p\left( \frac{d}{dt} \right) \ell' \right) dt =: Q_{K}(\ell)(0)$$

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Highest power of $\zeta$ and $\eta$ in $K$ is $n - 1$

$\implies Q_{K}$ is quadratic function of $\frac{d^i \ell}{dt^j}$, $j = 0, \ldots, n-1$

$Q_{K}$ is quadratic function of the state:
for every state map $X(\frac{d}{dt})$ there exists $K_X$ such that

$$Q_{K}(\ell) = \left( X\left( \frac{d}{dt} \right) \ell \right)^\top K_X \left( X\left( \frac{d}{dt} \right) \ell \right)$$
The observability Gramian $W$

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u \]

\[ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell \]

where $\text{GCD}(p, q) = 1$, $p$ stable, $\deg(q) \leq \deg(p)$
The observability Gramian $W$

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u \quad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell \]

where $GCD(p, q) = 1$, $p$ stable, $\deg(q) \leq \deg(p)$

In state-space framework, $W$ is defined as

\[ \int_{-\infty}^{0} y(t)^2 \, dt =: x_0^\top W x_0 \]

where $y$ is free response emanating from $x(0) = x_0$
The observability Gramian $W$

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u \]
\[ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell \]

where $\text{GCD}(p, q) = 1$, $p$ stable, $\deg(q) \leq \deg(p)$

In our framework: let $\ell \in C^\infty(\mathbb{R}, \mathbb{R})$. Then $W$ is

\[ Q_W(\ell)(0) := \int_0^{+\infty} \left( q\left(\frac{d}{dt}\right)\ell' \right) dt \]

where $\ell' \in C^\infty(\mathbb{R}_+, \mathbb{R})$ is such that

- $\ell'|_{(-\infty, 0]} = \ell|_{(-\infty, 0]}$
- $p\left(\frac{d}{dt}\right)\ell' = 0$ on $\mathbb{R}_+$
- $\left( q\left(\frac{d}{dt}\right)\ell', p\left(\frac{d}{dt}\right)\ell' \right) \in \mathcal{B}$
The observability Gramian $W$

\[ p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u \]

\[
\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q\left(\frac{d}{dt}\right) \\ p\left(\frac{d}{dt}\right) \end{bmatrix} \ell
\]

where $\text{GCD}(p, q) = 1$, $p$ stable, $\text{deg}(q) \leq \text{deg}(p)$

In our framework: let $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Then $Q_W$ is

\[
Q_W(\ell)(0) := \int_{0}^{\infty} \left( q\left(\frac{d}{dt}\right) \ell' \right) dt
\]

where $\ell' \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is such that

- $\ell'|_{(-\infty,0]} = \ell|_{(-\infty,0]}$
- $p\left(\frac{d}{dt}\right) \ell' = 0$ on $\mathbb{R}_+$
- $(q\left(\frac{d}{dt}\right) \ell', p\left(\frac{d}{dt}\right) \ell') \in \mathfrak{H}$

¿How to compute $W(\zeta, \eta)$?
Computation of $W(\zeta, \eta)$

\[
Q_W(\ell)(0) := \int_0^{+\infty} \left( q \left( \frac{d}{dt} \right) \ell' \right) dt
\]

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

\[
p(-\xi) f(\xi) + f(-\xi) p(\xi) = q(-\xi) q(\xi)
\]

Define $W$ from

\[
W(\zeta + \eta) = q(\zeta) q(\eta) - \left[ p(\zeta) f(\eta) + f(\zeta) p(\eta) \right]
\]
Computation of $W(\zeta, \eta)$

$$Q_W(\ell)(0) := \int_0^{+\infty} \left( q\left( \frac{d}{dt} \ell \right) \right) dt$$

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

$$p(-\xi)f(\xi) + f(-\xi)p(\xi) = q(-\xi)q(\xi)$$
Computation of $W(\zeta, \eta)$

$$Q_W(\ell)(0) := \int_{0}^{+\infty} \left( q\left(\frac{d}{dt}\right)\ell' \right) dt$$

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

$$p(-\xi)f(\xi) + f(-\xi)p(\xi) = q(-\xi)q(\xi)$$

Define $W$ from

$$(\zeta + \eta)W(\zeta, \eta) = q(\zeta)q(\eta) - [p(\zeta)f(\eta) + f(\zeta)p(\eta)]$$
Computation of $W(\zeta, \eta)$

$$Q_w(\ell)(0) := \int_0^{+\infty} \left( q\left( \frac{d}{dt} \ell \right) \right) dt$$

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

$$p(-\xi)f(\xi) + f(-\xi)p(\xi) = q(-\xi)q(\xi)$$

Define $W$ from

$$(\zeta + \eta) W(\zeta, \eta) = q(\zeta)q(\eta) - [p(\zeta)f(\eta) + f(\zeta)p(\eta)]$$
Computation of $W(\zeta, \eta)$

$$Q_W(\ell)(0) := \int_{0}^{+\infty} \left( q\left(\frac{d}{dt}\right)\ell' \right) \, dt$$

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

$$p(-\xi)f(\xi) + f(-\xi)p(\xi) = q(-\xi)q(\xi)$$

Define $W$ from

$$(\zeta + \eta)W(\zeta, \eta) = q(\zeta)q(\eta) - [p(\zeta)f(\eta) + f(\zeta)p(\eta)]$$

then

$$Q_W(\ell)(0) = \int_{0}^{+\infty} \left( q\left(\frac{d}{dt}\right)\ell \right)^2 \, dt$$

for all $\ell \in \ker p\left(\frac{d}{dt}\right)$
Computation of $W(\zeta, \eta)$

$$Q_W(\ell)(0) := \int_0^{+\infty} \left( q\left(\frac{d}{dt}\ell\right) \right) dt$$

Since $p$ is Hurwitz, there exists solution $f \in \mathbb{R}[\xi]$ to

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$$(\zeta + \eta)W(\zeta, \eta) = q(\zeta)q(\eta) - [p(\zeta)f(\eta) + f(\zeta)p(\eta)]$$

$Q_W$ is quadratic function of the state: for every state map $X\left(\frac{d}{dt}\right)$ there exists $W_X$ such that

$$Q_W(\ell) = \left( X\left(\frac{d}{dt}\ell\right) \right)^\top W_X \left( X\left(\frac{d}{dt}\ell\right) \right)$$
Balanced state maps

State map $X(\frac{d}{dt})$ is balanced if
Balanced state maps

State map \( X \left( \frac{d}{dt} \right) \) is balanced if

- If \( \ell_k \) is such that \( X(\ell_k)(0) \) is the \( k \)-th canonical basis vector, then

\[
Q_K(\ell_k)(0) = \frac{1}{Q_W(\ell_k)(0)}
\]

‘difficult to reach \( \iff \) difficult to observe’
Balanced state maps

State map $X\left(\frac{d}{dt}\right)$ is balanced if

- If $\ell_k$ is such that $X(\ell_k)(0)$ is the $k$-th canonical basis vector, then

$$Q_K(\ell_k)(0) = \frac{1}{Q_W(\ell_k)(0)}$$

‘difficult to reach $\iff$ difficult to observe’

- $Q_W(\ell_1)(0) \geq Q_W(\ell_2)(0) \geq \ldots \geq Q_W(\ell_n)(0) > 0$

or equivalently

$$0 < Q_K(\ell_1)(0) \leq Q_K(\ell_2)(0) \leq \ldots \leq Q_K(\ell_n)(0)$$

‘first who contributes most’
Balancing with QDFs

**Linear algebra**  $\implies$ there is basis $\{x_i^b \in \mathbb{R}_{n-1}[\xi]\}_{i=1,...,n}$ and $\sigma_i \in \mathbb{R}$ such that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$ such that

$$W(\zeta, \eta) = \sum_{i=1}^{n} \sigma_i x_i^b(\zeta) x_i^b(\eta) \quad K(\zeta, \eta) = \sum_{i=1}^{n} \frac{1}{\sigma_i} x_i^b(\zeta) x_i^b(\eta)$$
Balancing with QDFs

**Linear algebra** \(\iff\) there is basis \(\{x^b_i \in \mathbb{R}_{n-1}[\xi]\}_{i=1,...,n}\) and \(\sigma_i \in \mathbb{R}\) such that \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n\) such that

\[
W(\zeta, \eta) = \sum_{i=1}^{n} \sigma_i x^b_i(\zeta)x^b_i(\eta) \quad K(\zeta, \eta) = \sum_{i=1}^{n} \frac{1}{\sigma_i} x^b_i(\zeta)x^b_i(\eta)
\]

\(\sigma_i \sim (\text{classical})\) **Hankel singular values**
Balancing with QDFs

Linear algebra \(\iff\) there is basis \(\{x_i^b \in \mathbb{R}_{n-1}[\xi]\}_{i=1,\ldots,n}\) and \(\sigma_i \in \mathbb{R}\) such that \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n\) such that

\[W(\zeta, \eta) = \sum_{i=1}^{n} \sigma_i x_i^b(\zeta) x_i^b(\eta)\]

\[K(\zeta, \eta) = \sum_{i=1}^{n} \frac{1}{\sigma_i} x_i^b(\zeta) x_i^b(\eta)\]

Then

\[X^b(\xi) := \text{col}(x_i^b(\xi))_{i=1,\ldots,n}\]

is balanced state map.
Balancing with QDFs

Linear algebra \(\iff\) there is basis \(\{x_i^b \in \mathbb{R}_{n-1}[\xi]\}_{i=1,...,n}\)
and \(\sigma_i \in \mathbb{R}\) such that \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n\) such that

\[
W(\zeta, \eta) = \sum_{i=1}^n \sigma_i x_i^b(\zeta) x_i^b(\eta) \quad K(\zeta, \eta) = \sum_{i=1}^n \frac{1}{\sigma_i} x_i^b(\zeta) x_i^b(\eta)
\]

Then

\[
X^b(\xi) := \text{col}(x_i^b(\xi))_{i=1,...,n}
\]

is balanced state map.

(Classical) balanced state space representation: solve

\[
\begin{bmatrix}
\xi X^b(\xi) \\
q(\xi)
\end{bmatrix} =
\begin{bmatrix}
A_b & B_b \\
C_b & D_b
\end{bmatrix}
\begin{bmatrix}
X^b(\xi) \\
p(\xi)
\end{bmatrix}
\]
Balancing with QDFs

**Linear algebra** \(\iff\) there is basis \(\{x_i^b \in \mathbb{R}_{n-1} [\xi]\}_{i=1,...,n}\) and \(\sigma_i \in \mathbb{R}\) such that \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n\) such that

\[
W(\zeta, \eta) = \sum_{i=1}^{n} \sigma_i x_i^b(\zeta) x_i^b(\eta) \quad K(\zeta, \eta) = \sum_{i=1}^{n} \frac{1}{\sigma_i} x_i^b(\zeta) x_i^b(\eta)
\]

Then

\[
X^b(\xi) := \text{col}(x_i^b(\xi))_{i=1,...,n}
\]

is balanced state map.

**Classical** balanced state space representation: solve

\[
\begin{bmatrix}
\xi X^b(\xi) \\
n(\xi)
\end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} X^b(\xi) \\
p(\xi)
\end{bmatrix}
\]

**Model reduction by balancing** follows
Summary

- Working with functionals at most natural level;
Summary

• Working with functionals at most natural level;

• Two-variable polynomial representation;
Summary

- Working with functionals at most natural level;
- Two-variable polynomial representation;
- Operations/properties in time domain
  \( \sim \) algebraic operations;
Summary

- Working with functionals at most natural level;
- Two-variable polynomial representation;
- Operations/properties in time domain \(\sim\) algebraic operations;
- Differentiation, integration, positivity;
Summary

- Working with functionals at most natural level;
- Two-variable polynomial representation;
- Operations/properties in time domain → algebraic operations;
- Differentiation, integration, positivity;
- Lyapunov theory, dissipativity, model reduction by balancing.