Motivations and Historical Perspective

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MTNS - July 5-9, 2010
Block 1: Foundations [10:30 - 12.30]:

- **Talk 1:** Motivation and historical perspective, G. Marro [10:30 - 11:00]
- **Talk 2:** Invariant subspaces, L. Ntogramatzidis [11:00 - 11:30]
- **Talk 3:** Controlled invariance and invariant zeros, D. Prattichizzo [11:30 - 12:00]
- **Talk 4:** Conditioned invariance and state observation, F. Morbidi [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- **Talk 5:** Stabilization and self-bounded subspaces, L. Ntogramatzidis [15:30 - 16:00]
- **Talk 6:** Disturbance decoupling problems, L. Ntogramatzidis [16:00 - 16:30]
- **Talk 7:** LQR and $H_2$ control problems, D. Prattichizzo [16:30 - 17:00]
- **Talk 8:** Spectral factorization and $H_2$-model following, F. Morbidi [17:00 - 17:30]
What the geometric approach is

The geometric approach is a collection of tools, properties and algorithms for the analysis and control of dynamic linear and nonlinear systems in a coordinate-free context.

Statements in terms of subspaces (or, more generally, sets) instead of matrices are compact and clean, and insight into their meaning is highly facilitated.

This mini-course is an attempt to present in simple terms the origin, motivation and growth of the geometric approach through almost four decades.
Outline

- The beginning
- The main tools
- The most important solved problems
Our first papers in English

Controlled and Conditioned Invariant Subspaces in Linear System Theory

G. Basile and G. Marro

Communicated by G. Leitmann

Abstract. The concept of invariance of a subspace under a linear transformation is strongly connected with controllability and observability of linear dynamical systems. In this paper, we define controlled and conditioned invariant subspaces as a generalization of the simple invariants, for the purpose of investigating some further structural properties of linear systems. Moreover, we prove some fundamental theorems on which the computation of the above-mentioned subspaces is based. Then, we give two examples of practical application of the previous concepts concerning the determination of the constant output and perfect output controllability subspaces.

On the Observability of Linear, Time-Invariant Systems with Unknown Inputs

G. Basile and G. Marro

Communicated by G. Leitmann
Our first papers in English

References

Independently

The following words can often be found in the literature: “the geometric approach was introduced independently by Basile and Marro and Wonham and Morse”. This is true. The following papers are usually cited:


Let us look at two interesting documents concerning that period.
The end of the period of independence (January 29, 1970)

University of Toronto

DEPARTMENT OF ELECTRICAL ENGINEERING

January 29, 1970

Prof. G. Marro
Computing & Servomechanism Center
University of Bologna
Bologna, Italy

Dear Professor Marro:

Recently Prof. M. AouA brought to my attention your interesting recent papers in J. Opt. Th. and Appl., 3(5) and 3(6), 1969. In these papers you have exploited a geometric approach which, independently, A.S. Morse and I have also found to be very powerful in certain problems of multivariable linear synthesis.

Our recent reports are enclosed for your interest, and I would, of course, greatly appreciate receiving any further material in this area which you may have available for distribution.

With best wishes,

Sincerely yours,

W. M. Wonham
Associate Professor

WONHAM: jd1
Encl:
Another significant letter received in that period

Dear Dr. Marro,

I have noted with interest that you and Dr. Bucolo have been pursuing a theory of linear systems closely related to that being developed by my colleagues and myself here (as well as Wonham and Morse). I would appreciate receiving reprints of your work in the area and am enclosing several of my own. I am particularly interested at the moment in various types of observability (with differentiators and observers) which do not require input knowledge.

I am looking forward to hearing from you.

Sincerely yours,

Leonard Silverman
The three books on the geometric approach

Wonham (1974, 1979, 1985) has the great merit of having widely publicized the geometric approach.

Basile and Marro (1992) emphasize duality and dual problems, that were neglected in Wonham’s book. Software for Matlab is included.

Trentelman, Stoorvogel and Hautus (2001) provide a bridge towards achieving minimal $H_2$ and $H_{\infty}$ norm solutions.
Outline

- The beginning
- **The main tools**
- The most important solved problems
The concept of trajectory in the state space

\[ u \text{ (input)} \rightarrow \sum \rightarrow y \text{ (output)} \]

\[ x \text{ (state)} \]

Continuous-time:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

the term \( Du(t) \) is called *feedthrough*. 
Very often in practice the discrete-time systems are produced by sampling continuous-time systems, for instance with ZOH (zero order hold) equivalence. In these case a positive sampling time is associated to the conversion process.
The Matlab Representations of LTI Systems

*Purely dynamic continuous-time systems*

\[
>> \text{sys}=\text{ss}(A,B,C,0);
\]

*Continuous-time systems with feedthrough*

\[
>> \text{sys}=\text{ss}(A,B,C,D);
\]

*Purely dynamic discrete-time systems*

\[
>> \text{sys1}=\text{ss}(A,B,C,0,\text{Tc}); \quad (\text{Tc} = -1 \text{ if a sampling time is not specified})
\]

*Discrete-time systems with feedthrough*

\[
>> \text{sys1}=\text{ss}(A,B,C,D,\text{Tc});
\]

*Conversion from continuous to discrete time with the ZOH equivalence*

\[
>> \text{sys1}=\text{c2d}(\text{sys},\text{Tc}); \quad (\text{Tc} > 0 - \text{in this case a sampling time is specified})
\]
The main tools

Some properties of systems expressed in geometric terms

\[ V^* \cap S^* = \{0\} \]

\[ V^* + S^* = \mathcal{X} \]

minimal \( \rho \) such that \( V^* + S_\rho = \mathcal{X} \)

\[ \mathcal{Z}(\Sigma) \in \mathbb{C}_g \]

where \( \mathbb{C}_g \) is the set of stable modes in the complex plane.

\[ V = \text{vstar(sys)}; \]
\[ S = \text{sstar(sys)}; \]
\[ r = \text{reldeg(sys)}; \]
\[ z = \text{gazero(sys)}; \]
The main tools

Systems with multiple sets of inputs and outputs

\[ h \rightarrow \Sigma \rightarrow e \]
\[ u \rightarrow \Sigma \rightarrow y \]

<table>
<thead>
<tr>
<th>h: disturbance</th>
</tr>
</thead>
<tbody>
<tr>
<td>e: controlled output</td>
</tr>
<tr>
<td>u: control input</td>
</tr>
<tr>
<td>y: measured output</td>
</tr>
</tbody>
</table>

The system equation are

\[
\begin{align*}
\dot{x}(t) &= A x(t) + H h(t) + B u(t) \\
e(t) &= E x(t) + D_1 u(t) \\
y(t) &= C x(t) + D_2 h(t)
\end{align*}
\]

or

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A & H & B \\
E & 0 & D_1 \\
C & D_2 & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
h(t) \\
u(t)
\end{bmatrix}
\]

or else, by using a popular notation that points out the existence of feedthroughs terms, we can denote the system with

\[
\Sigma =
\begin{bmatrix}
A & H & B \\
E & 0 & D_1 \\
C & D_2 & 0
\end{bmatrix}
\]
Duality

The *dual* of system $\Sigma : (A, B, C, D)$ is defined as $\Sigma^T : (A^T, C^T, B^T, D^T)$.

Consider the interconnection of systems shown above: the overall dual system is obtained by reversing the order of serially connected systems and interchanging branching points with summing junctions and vice versa:

$$
\Sigma = \begin{bmatrix}
A_1 & 0 & 0 & B_1 \\
0 & A_2 & 0 & B_2 \\
B_{3,1} C_1 & B_{3,2} C_2 & A_3 & 0 \\
0 & 0 & C_3 & 0 \\
\end{bmatrix}, \quad \Sigma^T = \begin{bmatrix}
A_1^T & 0 & C_1^T B_{3,1}^T & 0 \\
0 & A_2^T & C_2^T B_{3,2}^T & 0 \\
0 & 0 & A_3^T & C_3^T \\
B_1^T & B_2^T & 0 & 0 \\
\end{bmatrix}
$$
Outline

- The beginning
- The main tools
- The most important solved problems
The disturbance decoupling problem (1969-70)

\[ \begin{aligned}
    \dot{x}(t) &= Ax(t) + Bu(t) + Hh(t) \\
y(t) &= Cx(t)
\end{aligned} \]

Let \( \mathcal{B} = \text{im} B, \mathcal{H} = \text{im} H, \mathcal{C} = \ker C \).

It will be shown that the solution in geometric terms depends on \( A, B, \mathcal{H}, \mathcal{C} \).


G. Basile and G. Marro, “L’invarianza rispetto ai disturbi studiata nello spazio degli stati,” in *Rendiconti della LXX Riunione Annuale AEI*, paper 1.4.01, Rimini, Italy, 1969,
The disturbance decoupling problem (1969-70)

The solution of a synthesis problem with geometric techniques as a rule includes

- A structural condition
- A stabilizability condition
Other basic problems (primal and dual)

Measurable signal decoupling

Unknown-input observation


R. Laschi and G. Marro, “Alcune considerazioni sull’osservabilità dei sistemi dinamici con ingressi inaccessibili” in *Rendiconti della LXX Riunione Annuale AEI*, paper 1.1.06, Rimini, Italy, 1969,
A review of exact feedforward model following

Exact model following as a measurable disturbance decoupling problem with stability.

NOTE:
If $\Sigma_m$ consists of $p$ independent SISO systems connected in parallel, we achieve exact row-by-row decoupling \textit{at no cost}. 
The most important solved problems

From exact feedforward to exact feedback model following

\[
\Sigma_c \quad u \quad \Sigma \quad y
\]
\[
\Sigma_m \quad y_m
\]

Exact feedback model following.

\[
\Sigma_c \quad u \quad \Sigma \quad y
\]
\[
+ \quad \eta = 0
\]
\[
\Sigma_m \quad y_m
\]

A structurally equivalent connection.

\[
\Sigma_c \quad u \quad \Sigma \quad y
\]
\[
+ \quad \eta = 0
\]
\[
\Sigma_m \quad y_m
\]

Another structurally equivalent connection.

\[
\Sigma_c \quad u \quad \Sigma \quad y
\]
\[
+ \quad \eta = 0
\]
\[
\Sigma_m \quad y_m
\]

Exact feedback model following, possibly with row-by-row decoupling and multiple internal models.

G. Marro (Bologna, Italy)
The most important solved problems

Other basic problems (primal and dual)

\[ \sum_{c,1} \alpha_1 \quad \sum_{c,2} \alpha_2 \]

\[ \sum c_1 \quad y_1 \]

\[ \sum \quad u \]

\[ \sum c_2 \quad y_2 \]

\[ \sum^T \quad u_1 \quad \sum^T \quad \alpha_1 \]

\[ \sum^T \quad u_2 \quad \sum^T \quad \alpha_2 \]

The noninteracting control problem

The fault detection and isolation problem


The most important solved problems

Other basic problems

Disturbance Decoupling Problem with dynamic output feedback


The main criticism to which the geometric approach is subjected to is that exact solutions to the above problems cannot be implemented in practice because of uncertainties in parameters.

In this approach standard optimal control and filtering problems can be treated in a minimal $H_2$ or $H_\infty$ norm context by simply substituting system $\Sigma$ with the corresponding Hamiltonian system.

Thus the geometric approach provides insight and tools for the treatment of singular and cheap cases of optimal control.

Interesting results in this area are due to Stoorvogel, Chen, Saberi and Sannuti (1990-2000).
The most important solved problems

Spectral factorization and $H_2$-optimal model following

$$\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t) \\
y_1(t) &= C_1 x(t) + D_1 u(t)
\end{align*}$$

The Stoorvogel problem


```matlab
>> sys=ss(A,B,C,D);
>> [C1,D1]=stoor(A,B,C,D,[Tc]);
```
The most important solved problems

Spectral factorization and $H_2$-optimal model following

\[
\begin{align*}
\Phi & \quad \sum \quad y \\
\Phi & \quad \sum^* \quad \bar{y}
\end{align*}
\]

\[
\begin{align*}
\Phi & \quad \sum_1 \quad y_1 \\
\Phi & \quad \sum_1^* \quad \bar{y}
\end{align*}
\]

The spectral factorization problem (the spectrum $\Phi$ is the same)

\begin{verbatim}
>> sys=ss(A,B,C,D);
>> adjsys=ss(-A','-C','B','D');
(adjoint system)

>> sys1=ss(A,B,C1,D1);
>> adjsys1=ss(-A','-C1','B','D1');
\end{verbatim}

The most important solved problems

The multivariable regulator problem

\[ \Sigma_e \quad r \quad e \quad \Sigma_c \quad u \quad \Sigma \quad y \]

The multivariable regulator problem with internal model (1977-1987)


The most important solved problems

The multivariable regulator problem

Exact feedback model following (2002) 

\[ r + h \rightarrow \Sigma_c \rightarrow \Sigma \rightarrow d \rightarrow \Sigma_e \rightarrow y \]

\[ \Sigma_m \rightarrow y_m \]

H₂-optimal feedb. model following (2007)

\[ r + h \rightarrow \Sigma_c \rightarrow \Sigma \rightarrow d \rightarrow \Sigma_o \rightarrow y \]

\[ \Sigma_m \rightarrow y_m \rightarrow e \]

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The geometric approach has been developed for about forty years gradually, as new linear and nonlinear problems arose and were analyzed.

It provides a significant insight into systems and control, based on very few elementary tools, on which the overall theory is based.

Although the geometric tools are very simple and supported by exhaustive computational machinery, it is rather difficult to get a complete panorama of them, since their presentation in the literature by several authors is not uniform in style and has very often been covered in unnecessarily heavy mathematics.
A great friend

To conclude, let me remember a great friend, Giuseppe Basile, who worked with me on the above research and is no longer with us.

I recall, in particular, his intuition for new ideas and his lively enthusiasm in discussion for developing them.

Thank you!