Some Solutions for Practice Final 2

Observe : Solutions here may sometimes be a little sketchier than what is expected in the exam. Make sure you justify all your steps.

3. (a) We have $c'(t) = (0, -\sin(t), \cos(t))$ and $\|c'(t)\| = 1$. Hence

$$\int_{\mathbf{C}} f \; ds \;=\; \int_0^{\pi/2} \cos^2(t)\sin(t) \; 1 \; dt \;=\; -\frac{1}{3}\cos^3(t)|_{t=0}^{t=\pi/2} \;=\; -0^3/3 + (1)^3/3 = \frac{1}{3}.$$

(b) We have $c'(t) = (-\sin(t), \cos(t), 1)$ and hence

$$\int_{\mathbf{C}} F \cdot ds \;=\; \int_0^{2\pi} (-\sin(t), \cos(t), e^t) \cdot (-\sin(t), \cos(t), 1) \; dt \;=$$

$$= \int_0^{2\pi} 1 + e^t \; dt \;=\; 2\pi + e^{2\pi} - 1.$$

(c) One can guess that $F = \nabla f$ with $f(x, y, z) = x^2 + y^2 + e^{xz}$. (Or check that *curl* $F = 0$ and calculate $f(x, y, z) = \int_c F \cdot ds$ for any curve from $(0, 0, 0)$ to (x, y, z).) Hence

$$\int_{\mathbf{C}} F \cdot ds \;=\; f(c(2\pi)) - f(c(0)) \;=\; f(1, 0, 1) - f(1, 0, 0) \;=\; 1 + e - 1 - 1 \;=\; e - 1.$$

4. Observe that for given $\mathbf{F}$ we have $P(x, y) = x$ and $Q(x, y) = x + y$. Hence it follows from Green's theorem that

$$\int_{\mathbf{C}} F \cdot ds \;=\; \int\int_D \frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x) \; dx \; dy \;=\; \int\int_D 1 \; dx \; dy \;=\; area(D) \;=\; 9\pi,$$

as D is the disk of radius 3.

5. (a) This is the parametrization of part of the unit sphere via spherical coordinates, where now $\phi = u$ and $\theta = v$. By a theorem in class (or by direct computation) we obtain

$$T_u \times T_v \;=\; \sin u(\sin u \cos v, \sin u \sin v, \cos u) \;=\; \sin u \; \Phi(u, v),$$

where $\Phi(u, v)$ is the given parametrization. Observe that the normal vector points outwards. We now obtain

$$\int\int_S \mathbf{F} \cdot dS \;=\; \int_0^{2\pi} \int_0^{\pi/4} (1, 1, 1) \cdot (\sin u \cos v, \sin u \sin v, \cos u) \; \sin u \; du \; dv \;=$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \sin^2 u(\cos v + \sin v) \;+\; \sin u \cos u \; du \; dv.$$

Changing the order of integration, we see that the integral over the first summand is equal to 0. Hence the integral is equal to

$$= 2\pi \int_0^{\pi/4} \sin u \ \cos u \ du \ = \ \pi \ \sin^2 u|_{u=0}^{u=\pi/4} \ = \ \frac{\pi}{2}.$$

6. S can be parametrized by $\Phi(x, \theta) \ = \ (x, \frac{x}{2}\cos\theta, x\sin\theta)$. Now calculate the integral

$$\int\int_S \mathbf{F}\cdot dS \ = \ \int_0^{2\pi}\int_1^2 \frac{1}{2}(x - x^2) \ dx \ = \ \pi \ \int_1^2 x - x^2 \ dx \ = \ \frac{-5}{6}.$$

7. (b) The region is given by the parametrization

$$x \ = \ r\cos\theta, \quad y \ = \ r\sin\theta, \quad z = z,$$

with $0 \le z \le 2$, $0 \le \theta \le \pi/2$ and $0 \le r \le \sqrt{x}$. We can then calculate the integral

$$\int\int\int_W x \ dx \ dy \ dz \ = \ \int_0^2 \int_0^{\sqrt{z}} \int_0^{\pi/2} (r\cos\theta) \ r \ dr \ d\theta \ dz.$$

(c) The divergence of the given vector field is $\frac{\partial}{\partial x}(x^2/4) + \frac{\partial}{\partial y}(xy/2) + \frac{\partial}{\partial z}(1) \ = \ x$. Hence using the Gauss divergence theorem, the value of the surface integral to be calculated here is equal to the value of the integral in (b).

8. We calculate that

$$div \ \mathbf{F} = \frac{\partial}{\partial x}(x + yz^3) + \frac{\partial}{\partial y}(-y + \sinh(x^2)) + \frac{\partial}{\partial z}(z + x^{72}) \ = \ 1 - 1 + 1 \ = \ 1.$$

It follows from Gauss' divergence theorem that

$$\int\int_S \mathbf{F}\cdot dS \ = \ \int\int\int_W div \ \mathbf{F} \ dV \ = \ \int\int\int_W 1 \ dV \ = \ vol(W) \ = \ \frac{4}{3}\pi abc.$$

9. This is a hard problem. There are two key observations:
- The x and the y coordinates of the curve c are the same
- The projection of the curve $c(t)$ into the xz plane still is a closed curve. Let D be the two-dimensionial region inside the curve $\tilde{c}(t) = (2\sin t - \sin 2t, 2\cos t - \cos t)$.
- Let S be the surface given by the parametrization

$$\Phi(u, v) = (u, u, v), \quad (u, v) \in D.$$

Then we have

$$T_u \times T_v \ = (1, 1, 0) \times (0, 0, 1) \ = \ (1, -1, 0),$$

and the boundary of S is given by the curve c. We can now apply Stokes' Theorem

$$\int_c F \cdot ds \ = \ \int\int_S curl \ F \ \cdot dS \ =$$

$$= \ \int\int_D (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cdot (1, -1, 0) \ du \ dv \ = \ \int\int_D 0 \ du \ dv \ = \ 0.$$

Observe that as the result is equal to 0, we did not have to worry about orientations or parametrization of D.