Determine the truth or falsity of each of the following statements. Be sure to explain your answers.

1. If $A$ is invertible and $AB$ is invertible, then $B$ is invertible.

*Solution.* This is true. If $A$ and $AB$ are invertible, then $A^{-1}$ is invertible and
\[ B = IB = (A^{-1}A)B = A^{-1}(AB). \]
Thus, as the product of invertible matrices is invertible, $B$ is invertible.

2. If $A$ is not invertible and $B$ is not invertible, then $AB$ is not invertible.

*Solution.* This is false. Let
\[ A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
Then neither $A$ nor $B$ is invertible, but
\[ AB = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2] \]
is invertible.

3. There exists a vector
\[ \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \]
such that $[T]$ is invertible, where $T$ is the linear function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula
\[ T(\vec{v}) = \vec{a} \times \vec{v}, \quad \vec{v} \in \mathbb{R}^3. \]

*Solution.* This is false. Choose a nonzero vector $\vec{x}$ in $\mathbb{R}^3$ that is colinear with $\vec{a}$. Then
\[ [T]\vec{x} = T(\vec{x}) = \vec{a} \times \vec{x} = 0. \]
But then $[T]$ cannot be invertible as it cannot have a left inverse: if $B[T] = I$, then
\[ \vec{x} = I\vec{x} = B[T]\vec{x} = 0, \]
contradicting the fact that $\vec{x}$ was chosen to be a nonzero vector.