Quiz 3 Solution

November 4, 2018

Use gaussian elimination to solve the following two problems involving the three vectors

\[ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

in \( \mathbb{R}^4 \).

1. Find all vectors \( \vec{w} \) in \( \mathbb{R}^4 \) that are orthogonal to \( \vec{v}_1 \), \( \vec{v}_2 \), and \( \vec{v}_3 \).

2. Determine the values for \( c \) such that

\[ \begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} \in \text{Span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}. \]

Solution to Problem 1. \( \vec{w} \) is orthogonal to \( \vec{v}_1 \), \( \vec{v}_2 \), and \( \vec{v}_3 \) if and only if

\[ \vec{v}_1 \cdot \vec{w} = 0, \quad \vec{v}_2 \cdot \vec{w} = 0, \quad \text{and} \quad \vec{v}_3 \cdot \vec{w} = 0, \]

or equivalently, the components of \( \vec{w} \) comprise a solution to the system

\begin{align*}
w_1 + w_2 + w_3 + 2w_4 &= 0 \\
w_1 + w_2 + 2w_3 + w_4 &= 0 \\
w_1 + 2w_2 + w_3 + w_4 &= 0.
\end{align*}

(1)

To solve this system we reduce the augmented matrix

\[ \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix} \]

by performing the following elementary row operations.

Step 1. Subtract the first row from the second row to obtain

\[ \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix} \]
Step 2. Subtract the first row from the third row to obtain
\[
\begin{bmatrix}
1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 \\
\end{bmatrix}.
\]

Step 3. Swap the second and third rows to obtain
\[
\begin{bmatrix}
1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 \\
\end{bmatrix}.
\]

Step 4. Subtract the second row from the first row to obtain
\[
\begin{bmatrix}
1 & 0 & 1 & 3 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
\end{bmatrix}.
\]

Step 5. Subtract the third row from the first row to obtain
\[
\begin{bmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
\end{bmatrix}.
\]

Thus, \(w_4\) is a nonpivotal variable and the general solution to (1) is given by choosing \(w_4\) to be an arbitrary real number and then defining \(w_1, w_2, \text{ and } w_3\) so that
\[
\begin{align*}
w_1 + 4w_4 &= 0 \\
w_2 - w_4 &= 0 \\
w_3 - w_4 &= 0.
\end{align*}
\]

Therefore the general vector \(\vec{w}\) orthogonal to \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\) has the form
\[
\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} -4w_4 \\ w_4 \\ w_4 \\ w_4 \end{bmatrix} = w_4 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

where \(w_4\) is an arbitrary real number.

Solution to Problem 2.
\[
\begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} \in \text{ Span } \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}
\]

if and only if there exist real numbers \(x_1, x_2, x_3\) such that
\[
\begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix},
\]

\[2\]
or equivalently, the system

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 1 \\
    x_1 + x_2 + 2x_3 &= c \\
    x_1 + 2x_2 + x_3 &= 1 \\
    2x_1 + x_2 + x_3 &= c
\end{align*}
\]  \hspace{1cm} (3)

is solvable. The augmented matrix associated with this system is

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & 2 & c \\
    1 & 2 & 1 & 1 \\
    2 & 1 & 1 & c
\end{bmatrix}
\]

which we reduce via the following row operations.

**Step 1.** Subtract the first row from the second and third rows and subtract 2 times the first row from the fourth row to obtain

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    0 & 0 & 1 & c - 1 \\
    0 & 1 & 0 & 0 \\
    0 & -1 & -1 & c - 2
\end{bmatrix}
\]

**Step 2.** Swap the second and third rows to obtain

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & c - 1 \\
    0 & -1 & -1 & c - 2
\end{bmatrix}
\]

**Step 3.** Subtract the second row from the first row and add the second row to the fourth row to obtain

\[
\begin{bmatrix}
    1 & 0 & 1 & 1 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & c - 1 \\
    0 & 0 & -1 & c - 2
\end{bmatrix}
\]

**Step 4.** Subtract the third row from the first row and add the third row to the fourth row to obtain

\[
\begin{bmatrix}
    1 & 0 & 0 & 2 - c \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & c - 1 \\
    0 & 0 & 0 & 2c - 3
\end{bmatrix}
\]

Now, the system (3) is solvable if and only if this last matrix does not have a pivot in the fourth row, i.e., \(2c - 3 = 0\). Therefore, (2) holds if and only if \(c = 3/2\).