Name: ______________________ PID: ______________________

Write your name and PID above, and your PID on subsequent pages. There are 6 questions, and 50 available points. You should not need any notes for this test. No calculators or other electronic devices are permitted. Remember to silence your phone. You should show enough working to convince us that you did the problems yourself.

1. (a) Find the slope of the line passing through the points $(-2, 3)$ and $(1, 4)$. $\boxed{3}$

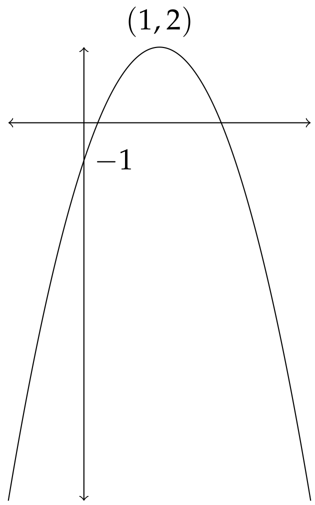
   Solution:
   $$\frac{4-3}{1-(-2)} = \frac{1}{3}$$

   (b) Find the equation of the line which passes through the point $(1, 0)$ and is perpendicular to the line from part (a). Write your answer in the form $y = mx + b$. $\boxed{3}$

   Solution: Every line perpendicular to the one in part (a) has slope $\frac{-1}{\frac{1}{3}} = -3$. The points (x, y) on the one passing through $(1, 0)$ are solutions of
   $$\frac{y-0}{x-1} = -3,$$
   which rearranges to
   $$y = -3x + 3.$$
   We can check this by plugging in $x = 1$, which should give $y = 0$.

2. Find the vertex of the parabola defined by $y = -3x^2 + 6x - 1$. Sketch the parabola, labeling the vertex and the y-intercept. $\boxed{5}$

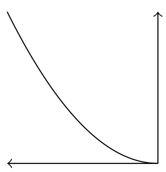
Solution: First rewrite y as $-3(x^2 - 2x + \frac{1}{3})$, to simplify completing the square. The coefficients of x^2 and x match with those of
$$(x-1)^2 = x^2 - 2x + 1.$$
Subtracting $\frac{2}{3}$ gives
$$(x-1)^2 - \frac{2}{3} = x^2 - 2x + \frac{1}{3}.$$
Therefore
$$y = -3\left((x-1)^2 - \frac{2}{3}\right) = -3(x-1)^2 + 2.$$
The largest possible value for y occurs when $x - 1 = 0$, i.e. $(x, y) = (1, 2)$.



3. Which of the following functions is one-to-one on the given domain? Illustrate your answers with a graph, using the horizontal line test where appropriate. $\boxed{4}$

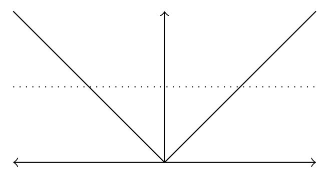
(a) $f(x) = x^2$ on $(-\infty, 0]$

Solution: This is one-to-one:



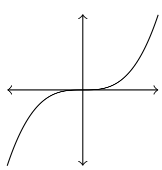
(b) $g(x) = |x|$ on $(-\infty, \infty)$

Solution: This is not one-to-one, as the dotted line passes the graph twice:



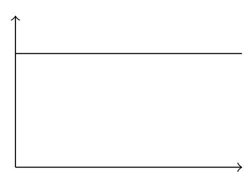
(c) $h(x) = x^3$ on $[-1, 1]$

Solution: This is one-to-one:



(d) $k(x) = 3$ on $(0, 3)$

Solution: This is not one-to-one, as the line is already horizontal, crossing itself at every point:



4. Define a function f by $f(x) = \sqrt{x}$ on $[0, \infty)$, and let g be given by $g(x) = f(2x - 4) + 1$.

(a) Write the domain of g as an interval. $\boxed{3}$

Solution: In order to make sense of $g(x)$, we just need $f(2x - 4) = \sqrt{2x - 4}$ to make sense. In other words $2x - 4 \geq 0$. This means $2x \geq 4$, or equivalently $x \geq 2$. As an interval, the domain of g is therefore $[2, \infty)$.

(b) Write the range of g as an interval. $\boxed{2}$

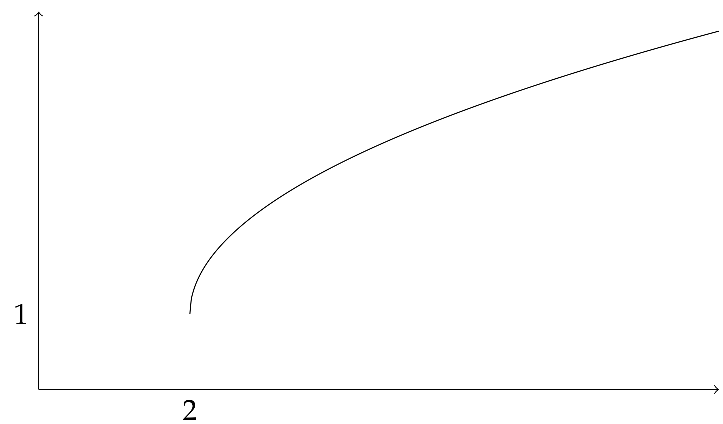
Solution: The range of f does not change under horizontal transformations, so we can ignore them for this question. There is a vertical transformation involved too, namely a shift up by 1. Since f has range $[0, \infty)$, the range of g is $[1, \infty)$.

(c) Describe a sequence of transformations (shifts or stretches, with directions and magnitudes) which take the graph of f to the graph of g. $\boxed{5}$

Solution: This has several answers. For example, shift up by 1, right by 4 and then shrink horizontally by a factor of 2. If you change the order of the right shift and the shrink, you have to shift right by 2 units instead. Apart from this, any order will work.

(d) Sketch the graph of g, labelling the endpoints of the domain and range (if any). $\boxed{3}$

Solution:



5. Define a function f by $f(x) = \frac{2}{3x-1}$.

(a) Write f as the composition of two (or more) simpler functions. $\boxed{4}$

Solution: This has multiple answers, for example $f = g \circ h \circ k$ where g is defined by

$$g(x) = \frac{2}{x}$$

and h is defined by

$$h(x) = x - 1$$

and k is defined by

$$k(x) = 3x.$$

(b) Find $f^{-1}(-2)$. You can do part (c) first if you're confident. $\boxed{4}$

Solution: If $x = f^{-1}(-2)$ then $\frac{2}{3x-1} = -2$, which means $3x - 1 = -1$ and hence $x = 0$.

(c) Find a formula expressing $f^{-1}(y)$ in terms of the number y. $\boxed{4}$

Solution: We can cheat and use part (a). Note that $g^{-1} = g$,

$$h^{-1}(y) = y + 1$$

and

$$k^{-1}(y) = \frac{y}{3}.$$

Therefore

$$f^{-1}(y) = k^{-1}(h^{-1}(g^{-1}(y))) = \frac{\frac{2}{y} + 1}{3} = \frac{2}{3y} + \frac{1}{3}.$$

Here is how to do it the 3C way: if $x = f^{-1}(y)$ then

$$y = f(x) = \frac{2}{3x - 1},$$

so

$$2 = y(3x - 1) = 3xy - y.$$

This means

$$3xy = 2 + y,$$

so

$$f^{-1}(y) = x = \frac{2 + y}{3y}.$$

This looks a bit different to the other answer, but they are the same. I don't really care which form you leave it in.

(d) Write the range of f^{-1} as an interval or union of intervals. $\boxed{2}$

Solution: The range of f^{-1} is the domain of f, which is the set of numbers such that $3x - 1 \neq 0$, i.e. $x \neq \frac{1}{3}$. As a union of intervals, the domain is $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

6. Define a function f by $\boxed{8}$

$$f(x) = \sqrt{|x - 2| - |x - 3|}.$$

Write the domain of f as an interval, a union of intervals, or a finite list of numbers.

Solution: The domain is the set of numbers x such that $|x - 2| - |x - 3| \geq 0$. Suppose $x \geq 3$. This ensures the expressions inside the absolute value signs are nonnegative, so

$$|x - 2| - |x - 3| = (x - 2) - (x - 3) = -2 + 3 = 1 \geq 0.$$

Therefore x is always in the domain in this case, so the domain contains $[3, \infty)$.

Now let's see what happens when $2 \leq x < 3$. In this case $x - 3 < 0$, so

$$|x - 2| - |x - 3| = (x - 2) - (-(x - 3)) = x - 2 + x - 3 = 2x - 5.$$

Such an x belongs to the domain when $2x - 5 \geq 0$, i.e. $x \geq \frac{5}{2}$. Thus the domain contains $[\frac{5}{2}, 3)$.

The last case is when $x < 2$. In this case both expressions are negative, so

$$|x-2| - |x-3| = -(x-2) - (-(x-3)) = 2 - x + x - 3 = -1.$$

Therefore such an x is never in the domain.

The final answer is $[3, \infty) \cup [\frac{5}{2}, 3) = [\frac{5}{2}, \infty)$.