Name: ____________________ PID: ____________________

Write your name and PID above, and your PID on subsequent pages. There are 6 questions, and 0 available points. You should not need any notes for this test. No calculators or other electronic devices are permitted. Remember to silence your phone. You should show enough working to convince us that you did the problems yourself.

1. Find the radius, center and circumference of the circle defined by $x^2 - 8x + y^2 + 2y = -14$.

> **Solution:** See 35 in §2.3 (solution on page 95).

2. Solve the following equation for x:

$$|x-3| + |x-4| = 1.$$

Write your answer as an interval, a union of intervals, or a finite list of x values.

> **Solution:** See 11 in §1.3 (solution on page 36).

3. (a) Find the equation of the line passing through the points $(3, -7)$ and $(5, -15)$. Write your answer in the form $y = mx + b$.

> **Solution:** By definition, the slope is
>
> $$m = \frac{-15 - (-7)}{5-3} = \frac{7-15}{2} = -\frac{8}{2} = -4.$$
>
> Therefore every point (x, y) on the line satisfies
>
> $$-4 = m = \frac{y - (-7)}{x-3} = \frac{y+7}{x-3}.$$
>
> Multiplying both sides by $x - 3$ gives
>
> $$-4x + 12 = y + 7,$$
>
> and hence every point on the line satisfies the equation
>
> $$y = -4x + 5.$$
>
> Conversely, if $x \neq 3$ then we can reverse the above steps to show that $(x, -4x+5)$ lies on the line. It is probably also a good idea to check that $-4x + 5 = -15$ when $x = 5$.

(b) Find a number t such that $(t, 2t)$ belongs to the line from part (a).

Solution: See 27 in §2.2 (solution on page 72).

(c) What would be the slope of a line perpendicular to the line from part (a)?

Solution: $\frac{-1}{-4} = \frac{1}{4}$

4. For each of the following equations, is there a function f such that every solution (x, y) of the equation satisfies $y = f(x)$? If so, give a formula for the function and write its domain as an interval or union of intervals.

(a) $y = x^2$

Solution: Yes. Such a function can be defined by $f(x) = x^2$ on $(-\infty, \infty)$.

(b) $y^2 = x$

Solution: No, because $(1, 1)$ and $(1, -1)$ are both solutions to $y^2 = x$, but only one y can satisfy the equation $y = f(1)$. Alternatively, the graph of $y^2 = x$ looks like a sideways parabola, and the vertical line $x = 1$ cuts it twice.

(c) $|y| = x$

Solution: No, because $(1, 1)$ and $(1, -1)$ are both solutions to $|y| = x$, but only one y can satisfy the equation $y = f(1)$. Alternatively, the graph of $y^2 = x$ looks like a $<$ sign, and the vertical line $x = 1$ cuts it twice.

(d) $xy = 1$

Solution: Yes. Such a function can be defined by $f(x) = \frac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$.

5. Define a function f by $f(x) = x^2$, and let g be given by $g(x) = 2f(\frac{x}{2} + 3)$.

(a) Give a list of transformations (shifts or stretches, either vertical or horizontal) which take the graph of f to the graph of g.

Solution: First, stretch vertically by a factor of 2. Then shift horizontally by -3. Finally, stretch horizontally by a factor of 2.

Note that the vertical stretch can be done at any time. However, if you try to stretch horizontally before shifting, you would instead have to shift by a "stretched" value of -6. One reason for this is that the x-intercept of g has to be at $x = -6$.

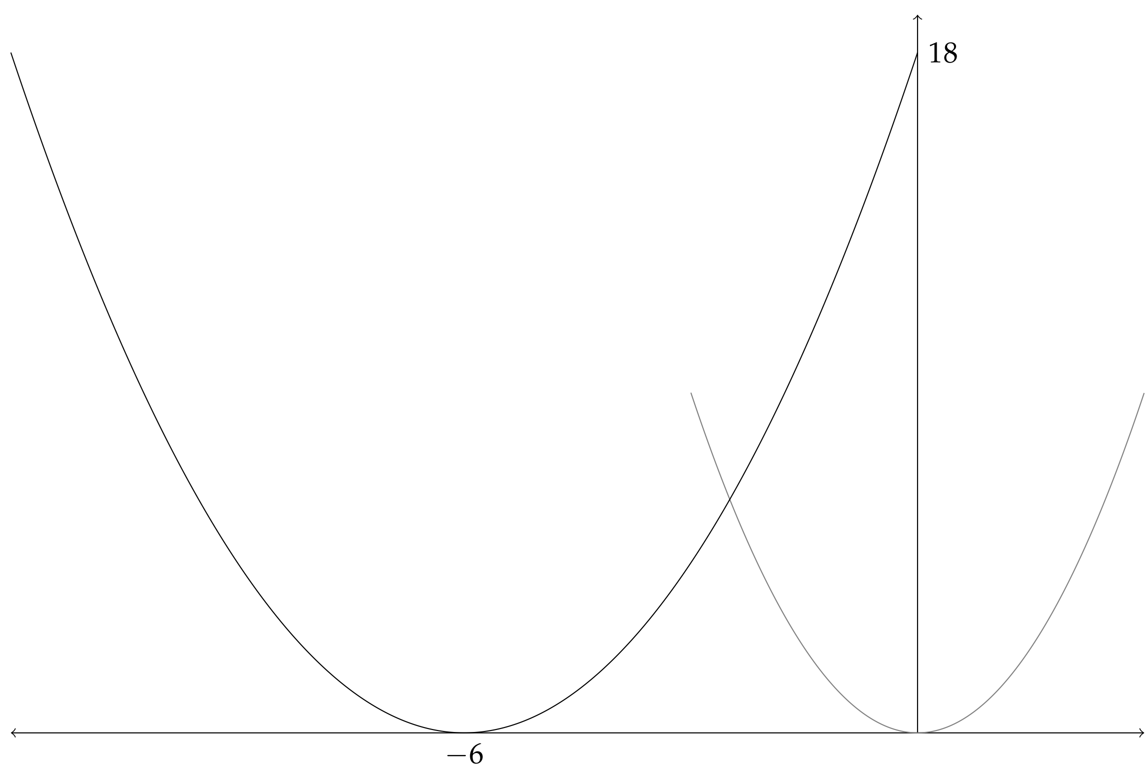
(b) Define functions h and k so that $g = h \circ f \circ k$.

Solution: One way is to put $h(x) = 2x$ and $k(x) = \frac{x}{2} + 3$.

Note that k can be decomposed further into $k_1(x) = \frac{x}{2}$ and $k_2(x) = x + 3$. The fact that $k = k_2 \circ k_1$ but $k \neq k_1 \circ k_2$ is the reason why the order matters for part (a). Shifting f is what gives $f \circ k_2$; stretching the latter then gives $(f \circ k_2) \circ k_1 = f \circ k$ which is one step away from g.

(c) Sketch the graph of g, labeling where it meets the x-axis and the y-axis.

Solution: In case it helps, I also included the graph of f in gray. I also scaled the whole thing vertically by $\frac{1}{2}$ to fit on the page.



6. Define a function f by $f(x) = \sqrt{x-2} + 3$. For (a), (b), (e) and (f), write your answer as an interval or union of intervals.

(a) What is the domain of f?

Solution: It is the set of all numbers x such that $x - 2 \geq 0$, or equivalently $x \geq 2$. This is the interval $[2, \infty)$.

(b) What is the range of f?

Solution: We know that the range of the square root function is $[0, \infty)$. To obtain f from this function, we shift the graph up by 3 and to the right by 2. The vertical shift changes the range to $[3, \infty)$, and the horizontal shift leaves this unchanged.

Here is an alternative approach avoiding transformations. Since $\sqrt{x-2} \geq 0$ for any x in the domain of f, we know that $f(x) \geq 3$. Conversely, if $y \geq 3$ then $y - 3 \geq 0$, so

$$y - 3 = \sqrt{(y-3)^2} = \sqrt{(y-3)^2 + 2 - 2}$$

and hence

$$y = f((y-3)^2 + 2).$$

In particular y belongs to the range of f. So the range is the set of numbers y such that $y \geq 3$, which we write as $[3, \infty)$.

A third approach (similar to the second) is to do part (e) first.

(c) What is $f^{-1}(3)$?

Solution: I hope it is not hard to read off that $f^{-1}(3) = 2$.

If not, here are the details. We need to find an x such that $f(x) = 3$. In other words

$$\sqrt{x-2} + 3 = 3.$$

Such an x satisfies $\sqrt{x-2} = 0$, and hence $x - 2 = 0$, giving $x = 2$.

(d) Find a formula expressing $f^{-1}(y)$ in terms of the number y.

Solution: Given y, we need to find an $x \geq 2$ such that $f(x) = y$. In other words

$$\sqrt{x-2} + 3 = y.$$

This means that

$$\sqrt{x-2} = y - 3,$$

which (since $x \geq 2$) implies that

$$x - 2 = |x - 2| = (\sqrt{x-2})^2 = (y-3)^2.$$

Therefore $f^{-1}(y) = (y-3)^2 + 2$.

(e) What is the domain of f^{-1}?

Solution: This one is a little tricky, because the formula in part (d) makes sense for any number y. However, f^{-1} itself is only defined on the range of f, which was $[3, \infty)$ as we saw in part (b).

(f) What is the range of f^{-1}?

Solution: This is just the domain of f, which is $[2, \infty)$ as we saw in part (a).