Name: ____________________ PID: ____________________

Write your name and PID above, and your PID on subsequent pages. There are 6 questions, and 50 available points. You may use a single page of notes. No calculators or other electronic devices are permitted. Remember to silence your phone. You should show enough working to convince us that you did the problems yourself.

1. (a) What is $\log(100)$? $\boxed{1}$

   Solution: $\log(100) = \log(10^2) = 2$.

   (b) What is $\log_2(8)$? $\boxed{1}$

   Solution: $\log_2(8) = \log_2(2^3) = 3$.

   (c) Let f be the function defined by $f(x) = \ln(x) - \ln(2x+1)$. Find a formula for f^{-1}. $\boxed{6}$

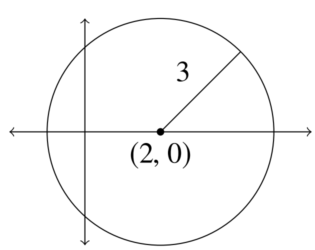
   Solution: If $y := f(x)$, then
$$\ln\left(\frac{x}{2x+1}\right) = y,$$
   so
$$\frac{x}{2x+1} = e^y$$
   and hence
$$x = e^y(2x+1) = 2xe^y + e^y.$$
   It follows that
$$e^y = x - 2xe^y = x(1-2e^y),$$
   so
$$f^{-1}(y) = x = \frac{e^y}{1-2e^y} = \frac{1}{e^{-y}-2}.$$

   (d) Write the range of f^{-1} as an interval. $\boxed{3}$

   Solution: The range of f^{-1} is the domain of f, which is
$$\{x \mid x > 0 \text{ and } 2x+1 > 0\} = \{x \mid x > 0\} = (0, \infty).$$

2. Sketch the graph of the circle defined by $x^2 - 4x + y^2 = 5$, labelling its exact center and radius. $\boxed{5}$

Solution: Since $x^2 - 4x = (x-2)^2 - 4$, the equation is $(x-2)^2 - 4 + y^2 = 5$. In other words $(x-2)^2 + y^2 = 9 = 3^2$, so the circle has center $(2, 0)$ and radius 3.



3. The polynomial $p(x) := x^3 - 3x^2 - 6x + 8$ has a zero at $x = 1$. Find the other zeros of $p(x)$. $\boxed{10}$

Solution: Dividing $p(x)$ by $x - 1$ gives

$$p(x) = (x-1)x^2 + x^2 \; - \; 3x^2 - 6x + 8 = (x-1)x^2 \; - \; 2x^2 - 6x + 8,$$

then

$$-2x^2 - 6x + 8 = (x-1)(-2x) - 2x \; - \; 6x + 8 = (x-1)(-2x) \; - \; 8x + 8,$$

and

$$-8x + 8 = (x-1)(-8),$$

so

$$p(x) = (x-1)x^2 + (x-1)(-2x) + (x-1)(-8) = (x-1)(x^2 - 2x - 8).$$

The other zeros of $p(x)$ are the zeros of $x^2 - 2x - 8$, namely

$$x = \frac{2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 6}{2} = 4 \text{ or } -2.$$

You might want to check your answer by computing that

$$p(4) = 4 \times 16 - 3 \times 16 - 6 \times 4 + 8 = 16 - 24 + 8 = 0$$

and

$$p(-2) = -8 - 3 \times 4 + 6 \times 2 + 8 = 0.$$

4. Ann has \$37 in her wallet, made from a mixture of \$2 and \$5 banknotes. If she has 11 notes in total, how many \$2 notes does she have? $\boxed{5}$

Solution: Let t and f be the number of \$2 and \$5 notes respectively. We know that

$$t + f = 11 \tag{1}$$

and

$$2t + 5f = 37. \tag{2}$$

Subtracting (1) from (2) twice gives

$$3f = 15,$$

so $f = 5$ and hence $t = 11 - f = 6$.

5. For this question, try to simplify your answers as much as you can, but you may leave large exponents, logs and roots unevaluated, e.g. $\sqrt{e}\log(33\pi) - 18^{41}$ is fine.

   (a) Ann puts her \$37 in a savings account with an interest rate of 6% per year, compounded every month. If she leaves the account alone, what will the balance be after 10 years? $\boxed{5}$

   Solution: The interest rate is 0.5% per month, and 10 years is 120 months, so the eventual balance will be

$$37(1.005)^{120}.$$

   (b) Another bank has savings accounts which pay continuously componded interest. What is the lowest interest rate it could offer Ann to make it worth her while to switch? $\boxed{6}$

   Solution: We need to find the rate r for which continuously compounded interest matches 6% per year compounded monthly. In other words $e^{rt} = (1.005)^{12t}$ for all t, and in particular $e^{r} = (1.005)^{12}$. Therefore

$$r = 12\ln(1.005).$$

6. Ann's acquaintance Jean-Ralphio brags to her that the balance of his investment account doubled in just one year.

   (a) Looking into it, Ann finds that Jean-Ralphio's balance b after t years is modelled by the formula $\boxed{4}$

   $$b(t) = \frac{6t+2}{t^2+t+2}.$$

   Assuming his balance continues to agree with this formula for several years, should Jean-Ralphio change his investment strategy? Why or why not?

   Solution: He should change his strategy, since when t is large,

   $$b(t) \approx \frac{6t}{t^2} = \frac{6}{t}$$

   is small, eventually becoming smaller than the original investment.

   (b) Hearing about Ann's doubts, Jean-Ralphio claims that his balance after t years is actually given by $\boxed{4}$

   $$a(t) = \frac{3+t}{3-t}.$$

   Explain why this is not realistic, with reference to the graph of a over the first few years.

   Solution: The formula is undefined when $t = 3$, and the graph has a vertical asymptote at 3, which is unrealistic because $b(t)$ will become very large when t is just below 3 years, but will suddenly be very negative after 3 years.