Name: ______________________ PID: ______________________

Write your name and PID above, and your PID on subsequent pages. There are 6 questions, and 50 available points. You may use a single page of notes. No calculators or other electronic devices are permitted. Remember to silence your phone. You should show enough working to convince us that you did the problems yourself.

1. (a) [§5.2] What is $10^{\log(7)}$? $\boxed{1}$

   Solution: By definition log is the inverse of the function f defined by $f(x) = 10^x$, so $10^{\log(7)}$ is just 7.

   (b) [§6.1] What is $\ln(e)$? $\boxed{1}$

   Solution: Note that $\ln(e) = \log_e(e) = \log_e(e^1)$, which is 1 by the argument from part (a).

   (c) [§5.2] Suppose x and y are numbers satisfying the equation $\boxed{5}$

$$3\log\left((y-4)^x\right) = 6,$$

   with $y > 4$. Give a formula for y in terms of x.

   Solution: Dividing by 3 gives

$$2 = \log\left((y-4)^x\right) = x\log(y-4).$$

   We know $x \neq 0$ because $\log\left((y-4)^0\right) = \log(1) = 0$, so

$$\log(y-4) = \frac{2}{x}$$

   and hence

$$y - 4 = 10^{\frac{2}{x}}.$$

   Finally $y = 10^{\frac{2}{x}} + 4$.

   (d) [§5.2] Which part of your argument breaks down when $y \leq 4$? $\boxed{2}$

   Solution: The right hand side of the equation $\log\left((y-4)^x\right) = x\log(y-4)$ does not make sense when $y-4 \leq 0$. This leads to additional solutions (e.g. $(x, y) = (2, -6)$) for which $y \neq 10^{\frac{2}{x}} + 4$.

2. [§2.3, §3.1] Define a function f by $f(x) = 2x^2 - 4x + 7$. Write the range of f as an interval. $\boxed{5}$

Solution: Since $x^2 - 2x = (x-1)^2 - 1$,

$$f(x) = 2(x-1)^2 - 2 + 7 = 2(x-1)^2 + 5.$$

This means that the range of f is $[5, \infty)$ (shifting the range of $g(x) = x^2$ up by 5).

3. (a) [§4.2] Let $p(x) := x^3 + x^2 - 2x$. Find a number r such that $p(x) - r$ is a polynomial multiple of $x - 3$. $\boxed{2}$

Solution: If $p(x) - r = (x-3)q(x)$ for some polynomial $q(x)$, then

$$p(3) - r = (3-3)q(3) = 0.$$

On the other hand

$$p(3) - r = 27 + 9 - 6 - r = 30 - r,$$

so $r = 30$.

(b) [§4.3] Find the zeros of the polynomial $q(x) := p(x) - r$ that you found in part (a). $\boxed{8}$

Solution: We know that 3 is one of the zeros. To find the others, we can divide $q(x)$ by $x-3$, as follows. First,

$$q(x) = x^3 + x^2 - 2x - 30.$$

Now

$$\frac{x^3}{x-3} = \frac{x^3 - 3x^2 + 3x^2}{x-3} = \frac{x^2(x-3) + 3x^2}{x-3} = x^2 + \frac{3x^2}{x-3},$$

which means

$$\frac{q(x)}{x-3} = x^2 + \frac{3x^2 + x^2 - 2x - 30}{x-3} = x^2 + \frac{4x^2 - 2x - 30}{x-3}.$$

Similarly

$$\frac{4x^2}{x-3} = 4x + \frac{12x}{x-3}$$

and hence

$$\frac{q(x)}{x-3} = x^2 + 4x + \frac{12x - 2x - 30}{x-3} = x^2 + 4x + \frac{10x - 30}{x-3} = x^2 + 4x + 10.$$

In other words $q(x) = (x-3)(x^2 + 4x + 10)$, so the other zeros of $q(x)$ are the zeros of $x^2 + 4x + 10$. But this quadratic polynomial has no zeros, because the discriminant $4^2 - 4 \times 10 = -24$ has no square root (in other words the quadratic formula makes no sense). Therefore 3 is the only zero of $q(x)$.

(c) [§4.2] Can you write $q(x)$ as a product of three linear (i.e. degree one) polynomials? $\boxed{2}$

Solution: No, because if $q(x)$ has factor of the form $x - a$, then a has to be a zero of $q(x)$, i.e. $a = 3$. But $q(x)$ is not a multiple of $(x-3)^3 = x^3 - 9x^2 + 27x - 27$.

4. (a) [§4.3] What happens to the graph of $\boxed{4}$

$$y = \frac{6x^6 - 7x^3 + 3}{3x^6 + 5x^4 + x^2 + 1}$$

when x is very large?

Solution: When x is really big, the lower degree terms don't matter much so

$$y \approx \frac{6x^6}{3x^6} = \frac{6}{3} = 2.$$

In other words, the graph has a horizontal asymptote along the line $y = 2$.

(b) [§4.3] Find the vertical asymptote of the graph of $\boxed{4}$

$$y = \frac{1 - 2x + 6x^2}{4x - 8}.$$

If you want y to be big, should you choose x to be on the left or right hand side of the asymptote?

Solution: The formula is undefined when $x = 2$, so the vertical asymptote is along the line $x = 2$.

When $x = 2$ the numerator is $1 - 4 + 24 = 21$, so when x is close to 2 we know that y is positive. The denominator is negative when $x < 2$, and positive when $x > 2$. Therefore y will be big when $x > 2$ is close to 2 (i.e. x is on the right hand side). On the left hand side y will be really negative.

You could also guess how to answer the second part by observing that

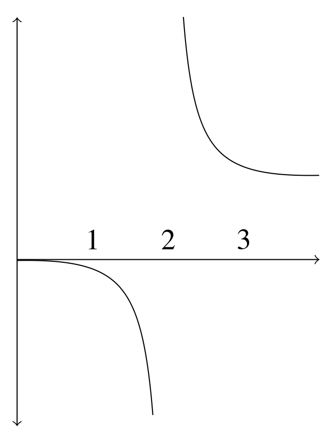
$$y = \frac{1 - 2 + 6}{4 - 8} = \frac{5}{-4} < 0$$

when $x = 1$, while

$$y = \frac{1 - 6 + 6 \times 9}{12 - 8} > \frac{6 \times 8}{4} = 12,$$

when $x = 3$.

If it helps, the graph looks like this:



Don't worry, I won't ask you to draw something like this without the help of a computer or calculator.

5. [§7.1, §7.2] A sample from a dinosaur bone is found to have 7 carbon atoms in it (both carbon-12 and carbon-14). These atoms are found to weigh a total of 88 atomic mass units. Assuming that carbon-n weighs n atomic mass units, how many carbon-12 atoms are in the sample? $\boxed{5}$

Solution: Let t and f be the number of carbon-12 and carbon-14 atoms respectively. We know that

$$t + f = 7 \tag{1}$$

and

$$12t + 14f = 88. \tag{2}$$

Multiplying (1) by 12 and subtracting the result from (2) gives

$$2f = 4,$$

so $f = 2$ and hence $t = 7 - f = 5$.

6. For this question, try to simplify your answers as much as possible, but you can leave logs and roots unevaluated, e.g. $\sqrt{e}\log(33\pi)$ is fine.

   (a) [§5.2, §5.3] A larger sample is taken from the bone and is found to have 1% of the carbon-14 it would have had when alive. Assuming the half-life of carbon-14 is 5730 years, how many years ago did the dinosaur pass away? $\boxed{3}$

   Solution: After t years, the proportion of carbon-14 remaining is

$$0.5^{t/5730} = 0.01,$$

   so

$$\log(0.01) = \log(0.5^{t/5730}) = \frac{t}{5730}\log(0.5),$$

   which means

$$t = 5730\frac{\log(0.01)}{\log(0.5)} = 5730 \times \frac{-\log(100)}{-\log(2)} = \frac{5730 \times 2}{\log(2)}.$$

   (b) [§5.2, §5.3] An audience member at the local archaeology conference insists that the bone is only 6000 years old. What would the half-life of carbon-14 have to be in order for the evidence to agree with this claim (given that 1% of the original carbon-14 remains in the bone)? $\boxed{4}$

   Solution: If h is the half-life in years, the proportion of carbon-14 remaining will be $0.5^{6000/h} = 0.01$, so

$$\log(0.01) = \log(0.5^{6000/h}) = \frac{6000}{h}\log(0.5).$$

   Therefore

$$h = 6000\frac{\log(0.5)}{\log(0.01)} = 6000\frac{-\log(2)}{-\log(100)} = 3000\log(2).$$

(c) [§6.3] The managers of Jurassic Park resurrect this dinosaur along with seven others. After a few years of observation they find that the dinosaur population is expanding exponentially at a continuous growth rate of 12% per year. How many dinosaurs will be in the park after 20 years? $\boxed{3}$

Solution: By the definition of continuous growth rate, after t years the population p is given by

$$p(t) = p(0)e^{0.12t} = 8e^{0.12t}.$$

In particular $p(20) = 8e^{2.4}$.