Name: ______________________ PID: ______________________

Write your name and PID above, and your PID on subsequent pages. There are 12 questions, and 100 available points. You may use a single page of notes. No calculators or other electronic devices are permitted. Remember to silence your phone. You should show enough working to convince us that you did the problems yourself. Unless otherwise specified, you can leave roots, logarithms and trigonometric functions unevaluated.

1. (a) Find an equation for the circle centered at $(0, -2)$ and passing through $(-4, 3)$. $\boxed{2}$

   Solution: [§2.1, §2.3] The radius of the circle should be the distance between the two points, which is

$$\sqrt{(0-(-4))^2+(-2-3)^2} = \sqrt{4^2+5^2} = \sqrt{16+25} = \sqrt{41}.$$

   So an equation for the circle is

$$x^2 + (y+2)^2 = 41.$$

   (b) Find the slope of the line passing through the points from part (a). $\boxed{2}$

   Solution: [§2.2]

$$\frac{-2-3}{0-(-4)} = -\frac{5}{4}.$$

   (c) Find the angle of the point $(-4, 3)$ on the circle from part (a) (measured in radians, counterclockwise from the positive horizontal direction). $\boxed{3}$

   Solution: [§10.1]

$$\tan^{-1}\left(-\frac{5}{4}\right) + \pi = \cos^{-1}\left(-\frac{4}{\sqrt{41}}\right) = \pi - \sin^{-1}\left(\frac{5}{\sqrt{41}}\right)$$

   (d) Find an equation for the line passing through the point $(-5, 0)$ which is perpendicular to the line from part (b). $\boxed{3}$

   Solution: [§2.2]

$$\frac{4}{5} = \frac{y}{x+5} \Rightarrow y = \frac{4}{5}(x+5) \Leftrightarrow y = \frac{4}{5}x + 4.$$

2. (a) Complete the square for $p(x) = -2x^2 + 8x + 10$, i.e. write it in the form $a(x+b)^2 + c$ where a, b and c are constants. $\boxed{6}$

   Solution: [§2.3] $(x-2)^2 = x^2 - 4x + 4$ so $-2(x-2)^2 = -2x^2 + 8x - 8$, which means $p(x) = -2(x-2)^2 + 18$.

   (b) Use part (a) to find the vertex of the parabola defined by $y = p(x)$. $\boxed{2}$

   Solution: [§2.3] The highest point on the parabola has x-coordinate 2, so it is $(2, p(2)) = (2, 18)$.

   (c) Using part (a), or another method, find the zeros of $p(x)$. $\boxed{2}$

   Solution: [§1.2] If $p(x) = 0$ then $-2(x-2)^2 = -18$, so $(x-2)^2 = \frac{18}{2} = 9$ and hence $x = 2 \pm \sqrt{9} = 2 \pm 3$ is either -1 or 5.

3. Using gaussian elimination, find all solutions to the system of equations $\boxed{9}$

$$x - y - 2z = 0$$

$$3x + 2y - 4z = -3$$
$$2x + 3y - 2z = 1.$$

Solution: [§7.2] Combining the first two equations tells us that

$$5y + 2z = (3x + 2y - 4z) - 3(x - y - 2z) = -3 - 3(0) = -3. \tag{1}$$

The first and last equations imply that

$$5y + 2z = (2x + 3y - 2z) - 2(x - y - 2z) = 1 - 2(0) = 1. \tag{2}$$

Together (1) and (2) give

$$0 = (5y + 2z) - (5y + 2z) = -3 - 1 = -4,$$

which is absurd, so the original system has no solutions!

4. Solve the inequality $|3x - 2| > |2x + 1|$ (express your answer using interval notation). $\boxed{9}$

Solution: [§1.3] If $x \geq \frac{2}{3}$ then $3x - 2 \geq 0$ and $2x + 1 \geq \frac{7}{3} > 0$, so

$$|3x - 2| - |2x + 1| = (3x - 2) - (2x + 1) = x - 3$$

is positive only when $x > 3$. On the other hand, if $x \leq -\frac{1}{2}$ then $2x + 1 \leq 0$ and $3x - 2 \leq -\frac{7}{2} < 0$, so

$$|3x - 2| - |2x + 1| = -(3x - 2) + (2x + 1) = -x + 3 \geq \frac{1}{2} + 3 > 0.$$

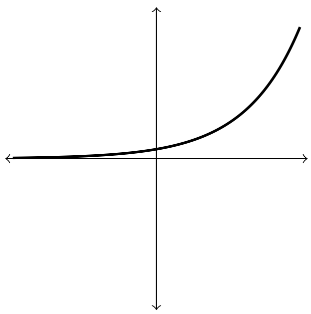
Finally, if $-\frac{1}{2} \leq x \leq \frac{2}{3}$ then $2x + 1 \geq 0$ but $3x - 2 \leq 0$, so

$$|3x - 2| - |2x + 1| = -(3x - 2) - (2x + 1) = -5x + 1$$

is positive only when $x < \frac{1}{5}$. Therefore $|3x - 2| > |2x + 1|$ if and only if $x > 3$ or $x < \frac{1}{5}$, i.e. the set of solutions is $(-\infty, \frac{1}{5}) \cup (3, \infty)$.

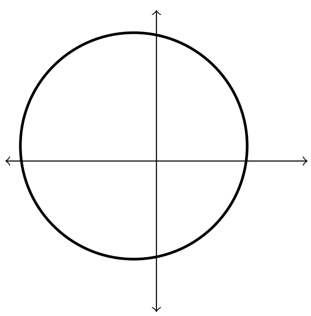
5. For each of the following graphs, state whether the graph defines a function, and if so, whether the function is one-to-one or not. $\boxed{8}$

(a)



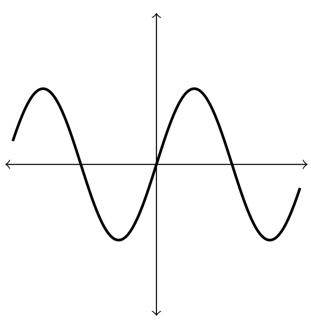
Solution: [§3.1, §3.4, §5.1] This is the graph of a one-to-one (exponential) function.

(b)



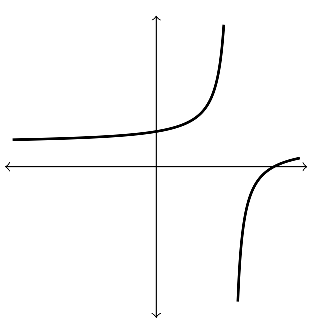
Solution: [§3.1] This is not the graph of a function, because it fails the vertical line test at (say) $x = 0$.

(c)



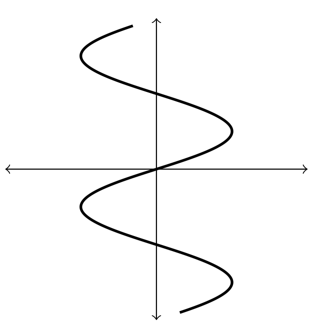
Solution: [§3.1, §3.4, §9.3] This is the graph of a function which is not one-to-one, because it fails the horizontal line test at (say) $y = 0$.

(d)



Solution: [§3.1, §3.4, §4.3] This is the graph of a one-to-one (rational) function.

(e)



Solution: [§3.1] This is not the graph of a function, because it fails the vertical line test at (say) $x = 0$.

6. Let f be the function defined by

$$f(x) = \ln\left(\frac{3}{2x-1}\right).$$

(a) Find the domain of f (express your answer using interval notation). $\boxed{3}$

Solution: [§3.1, §5.2, §6.1] The formula for f makes sense whenever $2x - 1 \neq 0$ and $\frac{3}{2x-1} > 0$. But $\frac{3}{2x-1} > 0$ if and only if $2x - 1 > 0$, or in other words $x > \frac{1}{2}$. In this case $2x - 1 \neq 0$ automatically, so the domain of f is $(\frac{1}{2}, \infty)$.

(b) Find a formula for f^{-1}. $\boxed{4}$

Solution: [§3.4, §6.1] $f^{-1}(y) = x$ whenever $y = f(x) = \ln\left(\frac{3}{2x-1}\right)$. In this case $e^y = \frac{3}{2x-1}$, so

$$2x - 1 = 3e^{-y}.$$

Therefore $x = \frac{1}{2}\left(3e^{-y} + 1\right)$, so $f^{-1}(y) = \frac{1}{2}\left(3e^{-y} + 1\right)$.

(c) Find the range of f (express your answer using interval notation). $\boxed{1}$

Solution: [§3.4, §5.1] The range of f is the domain of f^{-1}, which is the set $(-\infty, \infty)$ of all real numbers.

7. (a) Write $\frac{x}{x-3}$ as a polynomial plus a remainder of the form $\frac{r}{x-3}$. $\boxed{2}$

Solution: [§4.3]

$$\frac{x}{x-3} = \frac{(x-3)+3}{x-3} = 1 + \frac{3}{x-3}.$$

(b) Write $\frac{x^2}{x-3}$ as a polynomial plus a remainder of the form $\frac{r}{x-3}$. $\boxed{2}$

Solution: [§4.3]

$$\frac{x^2}{x-3} = \frac{(x^2-3x)+3x}{x-3} = x + 3\frac{x}{x-3}.$$

Using part (a) we can rewrite this as

$$x + 3\left(1 + \frac{3}{x-3}\right) = x + 3 + \frac{9}{x-3}.$$

(c) Write $\frac{x^3}{x-3}$ as a polynomial plus a remainder of the form $\frac{r}{x-3}$. $\boxed{2}$

Solution: [§4.3]

$$\frac{x^3}{x-3} = \frac{(x^3-3x^2)+3x^2}{x-3} = x^2 + 3\frac{x^2}{x-3}.$$

Using part (b) we can rewrite this as

$$x^2 + 3\left(x + 3 + \frac{9}{x-3}\right) = x^2 + 3x + 9 + \frac{27}{x-3}.$$

(d) Now write $\frac{x^3-2x^2-x-6}{x-3}$ as a polynomial (using the above, or otherwise). $\boxed{4}$

Solution: [§4.3]

$$\frac{x^3-2x^2-x-6}{x-3} = \left(x^2+3x+9+\frac{27}{x-3}\right) - 2\left(x+3+\frac{9}{x-3}\right) - \left(1+\frac{3}{x-3}\right) - \frac{6}{x-3} = x^2+x+2.$$

(e) Find the equation for any asymptotes of the graph $y = \frac{x-3}{x^3-2x^2-x-6}$ (the reciprocal of the above fraction). $\boxed{2}$

Solution: For large $|x|$ the corresponding y value is approximately $\frac{x}{x^3} = \frac{1}{x^2}$, which is close to zero. Therefore the graph has a horizontal asymptote along $y = 0$.

The denominator is only zero when $x = 3$, because x^2+x+2 has no zeros (by the quadratic formula). However, this is not an asymptote, because when $x \neq 3$

$$\frac{x-3}{x^3-2x^2-x-6} = \frac{1}{x^2+x+2},$$

so when x is close to 3, y is close to

$$\frac{1}{3^2+3+2} = \frac{1}{14}.$$

8. For this question, simplify your answers as much as you can. Your final answer should **not** involve exponents, logarithms or roots.

(a) Evaluate $\boxed{3}$

$$\log_3\left(\frac{1}{9^5}\right) - \log_3\left(\sqrt{3}\right).$$

Solution: [§5.2]

$$\log_3\left(\frac{1}{9^5}\right) - \log_3\left(\sqrt{3}\right) = \log_3\left(\left(3^2\right)^{-5}\right) - \log_3\left(3^{1/2}\right) = \log_3\left(3^{-10}\right) - \frac{1}{2} = -10 - \frac{1}{2} = -\frac{21}{2}.$$

(b) Solve the equation $\boxed{4}$

$$\log_2(x-3) + \log_2(x-2) = 1.$$

Solution: [§5.3, §2.3] Since

$$1 = \log_2(x-3) + \log_2(x-2) = \log_2\left((x-3)(x-2)\right),$$

we get

$$2 = 2^1 = (x-3)(x-2) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6.$$

Therefore

$$x^2 - 5x + 4 = 0,$$

which means

$$x = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1.$$

However $x = 1$ is not a solution because the equation makes no sense for $x \leq 3$. Therefore $x = 4$.

9. (a) A beekeeper observes that the population of a bee colony decreases by 10% every month. At the start of 2015 the population was 1234. Find the population at the start of 2016. [3]

Solution: [§5.3] The population $p(t)$ after t months is given by

$$p(t) = 1234(0.9)^t,$$

so after 12 months the population was $1234(0.9)^{12}$.

(b) The beekeeper makes some changes early in 2016, after which the bee population begins to grow exponentially. At the beginning of 2017 the population was 500, which grew to 1000 after one year. Find the **continuous** growth rate (per year) of the population during 2017. [3]

Solution: [§6.3] If r is the continuous growth rate then the population $q(t)$ after t years is

$$500e^{rt},$$

so $1000 = q(1) = 500e^r$. Therefore $e^r = \frac{1000}{500} = 2$, so $r = \ln(2)$, or $\ln(2) \times 100\%$ per year.

10. For this question, simplify your answers as much as you can. Your final answer should **not** involve any trigonometric functions.

(a) Convert $\frac{7\pi}{18}$ (radians) to degrees. [1]

Solution: [§9.2]

$$\frac{7\pi}{18} = \frac{7\pi}{18} \times \frac{180^\circ}{\pi} = 7 \times \frac{180^\circ}{18} = 70^\circ.$$

(b) Evaluate $\cos(240^\circ)$. [2]

Solution: [§9.3]

$$\cos(240^\circ) = \cos(180^\circ + 60^\circ) = -\cos(60^\circ) = -\frac{1}{2}.$$

(c) Suppose $\pi < \theta < \frac{3\pi}{2}$ and $\sin(\theta) = -\frac{1}{2}$. Evaluate $\tan(\theta)$. [3]

Solution: [§9.4] Since θ is in the given range $\cos(\theta) < 0$, and in fact

$$\cos(\theta) = -\sqrt{1 - \sin(\theta)^2} = -\frac{\sqrt{3}}{2}.$$

Therefore

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

(d) Evaluate $\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$. [2]

Solution: [§10.2] The points $(-\frac{3}{5}, y)$ on the unit circle satisfy

$$\left(\frac{3}{5}\right)^2 + y^2 = 1,$$

so

$$y = \pm\sqrt{1 - \frac{9}{25}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}.$$

The one corresponding to $\cos^{-1}(-\frac{3}{5})$ is $(-\frac{3}{5}, \frac{4}{5})$. Therefore $\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

11. A contractor is planning to build a ramp to help UCSD students reach the new train platform at Pepper Canyon. The platform is 20ft above the ground, and the ramp will be 160ft long. Find the angle of elevation of the ramp (in radians). $\boxed{3}$

Solution: [§10.1]

$$\sin^{-1}\left(\frac{20}{160}\right) = \sin^{-1}\left(\frac{1}{8}\right).$$

12. On a typical day in August the temperature high in San Diego was 77°F. The corresponding low was 67°F. You may assume that the highest and lowest temperatures occur at noon and midnight respectively.

(a) Find a function f of the form $f(t) = a\cos(bt) + c$ such that $f(t)$ models the temperature (in Fahrenheit) at t hours after the start of August (i.e. midnight between July 31 and August 1). $\boxed{7}$

Solution: [§11.2] The temperature does a full cycle in 24 hours, so if $t = 24$ then bt should be 2π. In other words $b = \frac{2\pi}{24} = \frac{\pi}{12}$. At the start, when $\cos(bt) = \cos(0) = 1$, the temperature should be 67°F, so $a + c = 67$. After 12 hours the temperature should be 77°F, while $\cos(bt) = \cos(\pi) = -1$, so $-a + c = 77$. It follows that $c = 72$ and $a = -5$, which means

$$f(t) = -5\cos\left(\frac{\pi}{12}t\right) + 72.$$

(b) Sketch the graph of the function you defined for part (a), labelling the first three lows and highs. $\boxed{3}$

Solution: [§11.2]