Finish limits at \( \infty \)

\[
\lim_{{x \to \infty}} \frac{f(x)}{g(x)} \quad \text{where } f(x) \text{ and } g(x) \text{ are polynomials}
\]

\[
\left( \text{i.e. } \lim_{{x \to \infty}} \frac{x^2 + 2x + 3}{-x^3 - 1} \right)
\]

- If \( \deg(f) > \deg(g) \), then \( \lim_{{x \to \infty}} \frac{f(x)}{g(x)} = \infty \) or \(-\infty\) depending on the sign of the leading coefficients of \( f \) and \( g \).
- If \( \deg(f) < \deg(g) \), then \( \lim_{{x \to \infty}} \frac{f(x)}{g(x)} = 0 \).
- If \( \deg(f) = \deg(g) \), then \( \lim_{{x \to \infty}} \frac{f(x)}{g(x)} = \frac{\text{leading term of } f(x)}{\text{leading term of } g(x)} \).
Rules

\(-\frac{\text{nonzero}}{0} = \pm \infty\)

\(-\frac{0}{\text{nonzero}} = 0\)

\(-\frac{\infty}{\text{nonzero}} = \pm \infty\)

\(-\frac{\pm \infty}{\text{nonzero}} = 0\)
2.4 Continuity

**Def.** A function \( f \) is continuous at a number \( a \) if
\[
\lim_{{x \to a}} f(x) = f(a).
\]

To check if \( f(x) \) is continuous at \( a \), one must check three things:

1. \( a \) is in the domain of \( f \) (so \( f(a) \) is defined).
2. \( \lim_{{x \to a}} f(x) \) exists
3. \( \lim_{{x \to a}} f(x) = f(a) \)

If any of (1), (2), or (3) are not satisfied, then \( f(x) \) is not continuous at \( a \). We say \( f \) has a discontinuity (or is discontinuous) at \( a \) if this is the case.
Geometric meaning: \( f(x) \) is continuous if the graph of \( f \) can be drawn without picking up your pencil.

The following types of functions are continuous at every number in their domain:

- Polynomials
- Root functions
- Exponential functions
- Rational functions
- Trigonometric functions
- Logarithmic functions

Examples of discontinuities

1) Where is \( f(x) = \frac{x-1}{x^2-1} \) continuous? Where are its discontinuities?

\( f(x) \) is a rational function, so \( f(x) \) is continuous on its domain.

Domain of \( f(x) \): \( x^2 - 1 = 0 \)

\( (x-1)(x+1) = 0 \)

\( x = 1 \) or \( x = -1 \)

The domain of \( f(x) \) is \( x \neq 1 \) and \( x \neq -1 \).
$f(x)$ is continuous at all $x$ except $x=1$ and $x=-1$.
The discontinuities of $f(x)$ are at $x=1$ and $x=-1$.

$$y = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$$

graph of $f$
\[ f(x) = \begin{cases} 
2x - 1 & \text{if } x \neq 2 \\
5 & \text{if } x = 2 
\end{cases} \]

where is \( f(x) \) continuous? where are discontinuities?

\( 2x - 1 \) and 5 are polynomials. So \( f(x) \) is continuous everywhere since it is a polynomial.

Since \( f(x) = 2x - 1 \) when \( x \neq 2 \), \( f(x) \) is continuous when \( x \neq 2 \).

Since \( x = 2 \) is a discrete point we cannot conclude \( f(x) \) is continuous at \( x = 2 \) because 5 is a polynomial.

\( f(x) \) is continuous everywhere except possibly at \( x = 2 \).

At \( x = 2 \)
\[ \lim_{x \to 2} f(x) \quad \frac{\text{because when } x \neq 2}{f(x) = 2x - 1} \quad \lim_{x \to 2} (2x - 1) = 2 \cdot 2 - 1 = 3 \]

\[ f(2) = 5 \]

\[ \lim_{x \to 2} f(x) = 3 \neq 5 = f(2) \quad \text{so } f(x) \text{ is not continuous at } x = 2. \]

Discontinuities of \( f \) are at \( x = 2 \). \( f(x) \) is continuous everywhere else.
\( f(x) = \begin{cases} 
2x - 1 & \text{if } x < 2 \\
x^2 & \text{if } x \geq 2 
\end{cases} \)

Where is \( f \) continuous, where are discontinuities?

The only place \( f(x) \) is possibly discontinuous is at \( x = 2 \).

At \( x = 2 \)

\[ \lim_{{x \to 2}} f(x) = \quad \text{to determine the limit we calculate} \]

\[ f(2) = 2^2 = 4 \]

\[ \lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} x^2 = 2^2 = 4 \]

\[ \lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} 2x - 1 = 2 \cdot 2 - 1 = 3 \]
\[
\lim_{x \to 2} f(x) \text{ does not exist because}
\]
\[
\lim_{x \to 2^-} f(x) = 3 \neq 4 = \lim_{x \to 2^+} f(x).
\]

Therefore, \( f(x) \) is not continuous at \( x = 2 \).
Intermediate Value Theorem (IVT)

Suppose $f$ is continuous on the closed interval $[a, b]$, and let $N$ be any number between $f(a)$ and $f(b)$ where $f(a) 
eq f(b)$. Then there exists a $c$ in $(a, b)$ such that $f(c) = N$.

Example (Using IVT to prove equations have solutions.)

(1) show that $e^x = 3 - 2x$ has a solution for $x$. 
Use IVT:

- Move everything to one side
  \[0 = 3 - 2x - e^x\]

- Set \(f(x) = 3 - 2x - e^x\)

- Now find \(x\) such that \(f(x) = 0\).

- Find \(a\) such that \(f(a) < 0\)

- \(b = \) \(f(b) > 0\)

- Then there exists by IVT a \(c\) such that \(f(c) = 0\). (\(N = 0\))

Try \(a = 0, \ b = 1\)

\[f(0) = 3 - 2\cdot0 - e^0 = 3 - 0 - 1 = 2 > 0\]

\[f(1) = 3 - 2\cdot1 - e^1 = 3 - 1 - 2.71828 < 0\]

By IVT there exists a \(c\) between 0 and 1 such that

\[f(c) = 0 / 3 - 2c - e^c = 0\]