

2.7 The Derivative as a Function

The derivative of f is the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of $f'(x)$ is all a such that $f'(a)$ exists and is a number.

Example Determine the derivative of $f(x) = x^3 - x$. Compare the graphs of $y = f(x)$ and $y = f'(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$$

$$\frac{(x+h)^3 - (x+h) - (x^3 - x)}{h} = \frac{\overbrace{(x+h)(x+h)(x+h)}^{(x+h)(x+h)(x+h)} - x - h - x^3 + x}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2 - 1)}{h} = 3x^2 + 3xh + h^2 - 1$$

Then

$$f'(x) = \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 - 1 = 3x^2 + 3x \cdot 0 + 0^2 - 1$$

$$= 3x^2 - 1$$

$$x^3 - x = x(x^2 - 1)$$

$$\parallel x(x-1)(x+1)$$

$$f'(x) = 3x^2 - 1$$

$$y = x^3 - x$$

$$y = f(x)$$

tangent lines
have positive
slope

tangent lines
have positive
slope

tangent lines have
negative slope

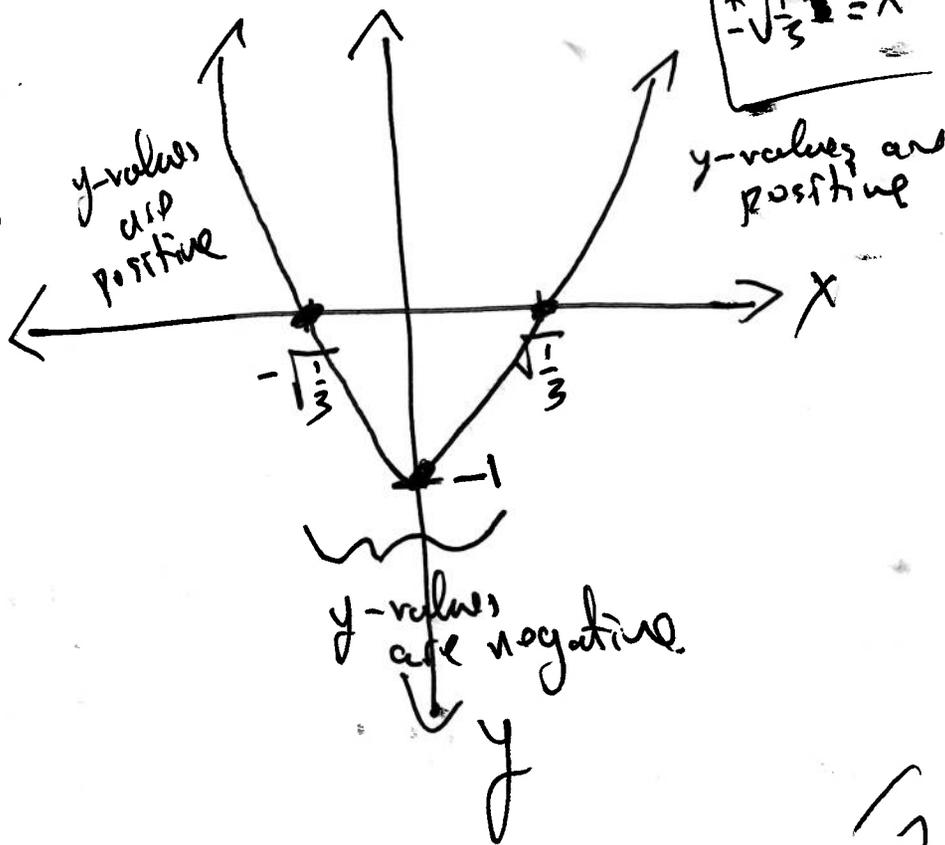
y-values are slopes of
tangent lines to
 $y = f(x)$
 $y = 3x^2 - 1$

$$0 = 3x^2 - 1$$

$$1 = 3x^2$$

$$\frac{1}{3} = x^2$$

$$\pm \sqrt{\frac{1}{3}} = x$$



Example Determine the derivative of $f(x) = \sqrt{x}$. Graph and compare.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

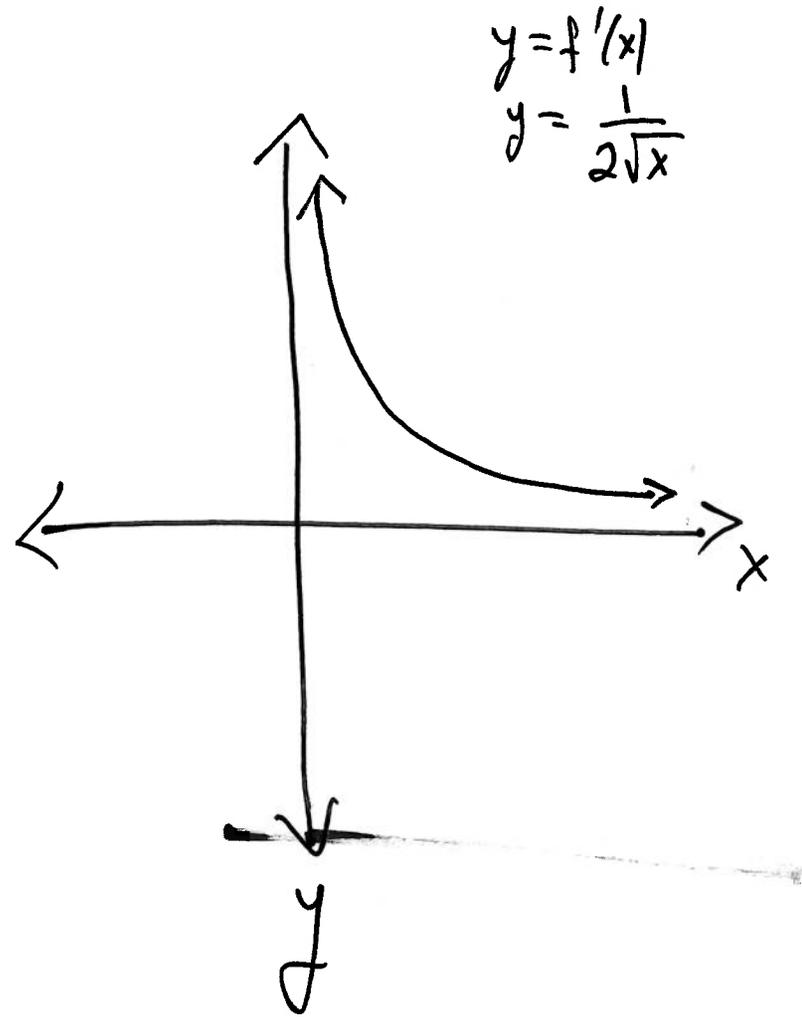
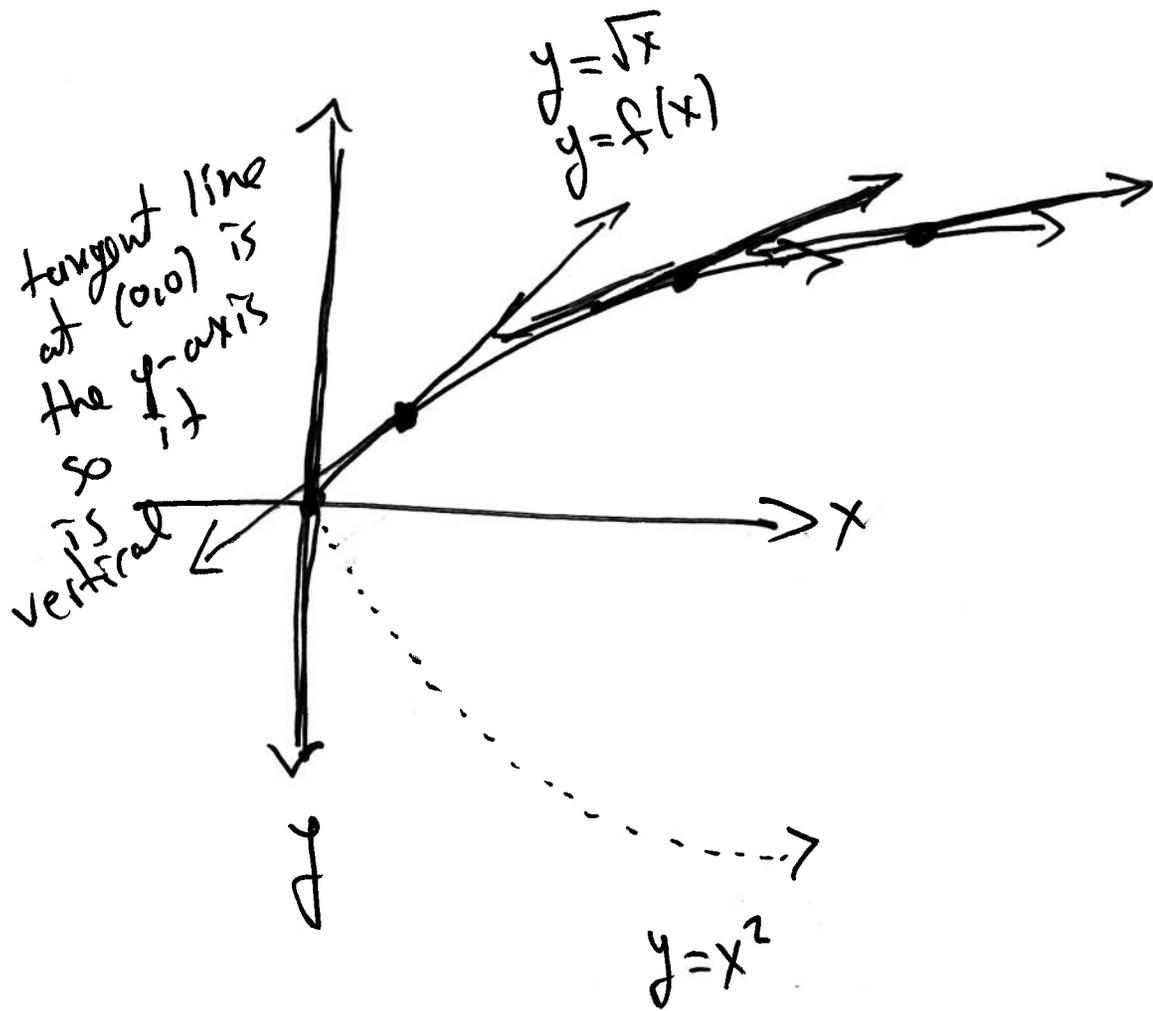
$$\frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}} = \frac{\sqrt{x+h}\sqrt{x+h} + \sqrt{x}\sqrt{x+h} - \sqrt{x}\sqrt{x+h} - \sqrt{x}\sqrt{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\text{Then } f'(x) = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\text{Then } \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$



Def: A function f is differentiable at $x=a$ if $f'(a)$ exists and is a number.

$$\left(f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right)$$

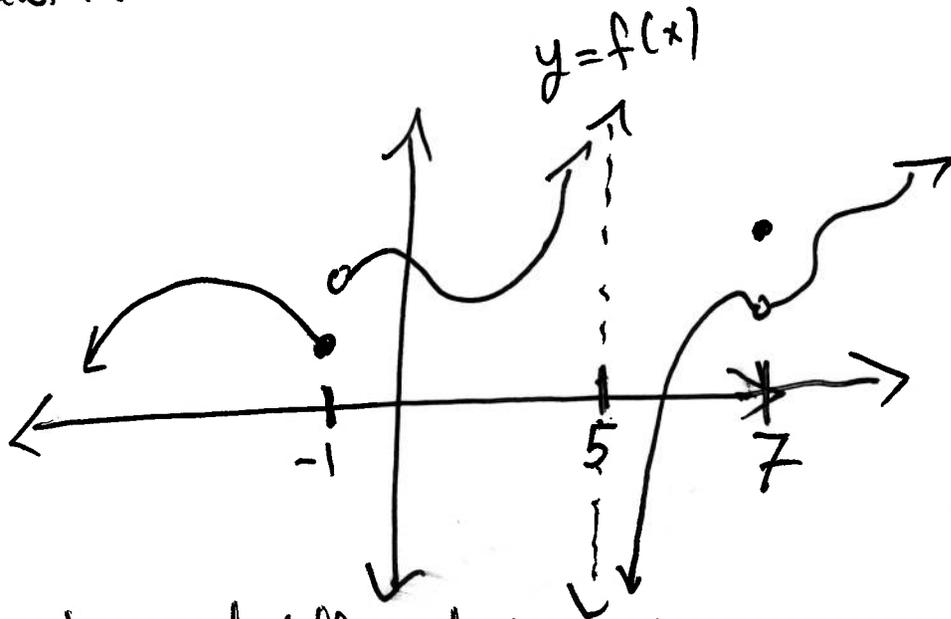
Thm: If f is differentiable at a , then f is continuous at a .

Differentiability is a stronger condition than continuity.

How can functions not be ~~continuous~~ differentiable

① If f is not continuous at a , then f is not differentiable.

EX

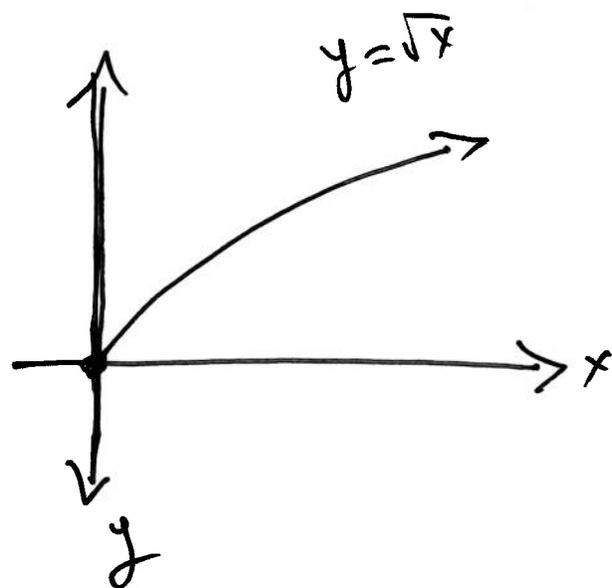


are some points
where $f(x)$ is not
differentiable?

$f(x)$ is not ~~not~~ differentiable at $x=-1$, $x=5$, and $x=7$ because
 $f(x)$ is not continuous there.

(2) $y = f(x)$ has a vertical tangent line at $x = a$.

EX ~~$f(x) = \sqrt{x}$~~



What is

$$\frac{\sqrt{h}}{h} = \frac{h^{1/2}}{h} = h^{1/2} \cdot h^{-1} = h^{1/2-1} = h^{-1/2} = \frac{1}{h^{1/2}} = \frac{1}{\sqrt{h}}$$

Is $f(x) = \sqrt{x}$ differentiable at $x = 0$?

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h} - \sqrt{0}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{h}}{h} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}}$$

$\lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty$ because when $h > 0$, $\sqrt{h} > 0$

$\lim_{h \rightarrow 0^-} \frac{1}{\sqrt{h}}$ does not exist because if $h < 0$, \sqrt{h} is not defined

Therefore ~~lim~~ $f'(0) = \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}}$ does not exist.

Then $f(x)$ is not differentiable at $x=0$.

③ ~~graph~~ $f(x)$ is not differentiable at $x=a$ if the graph $y=f(x)$ ~~is~~ is not smooth at $(a, f(a))$ or has a sharp edge or corner at $(a, f(a))$.

EX: $f(x) = |x|$

