

## Double Chain Rule

$f(x) = \sin(\cos(\tan(x)))$ , determine  $f'(x)$ .

$$y = \sin(\underbrace{\cos(\tan(x))}_{\text{inside}}), \quad y = \sin(u), \quad u = \cos(\tan(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \frac{dy}{du} = \cos(u) \quad \frac{du}{dx} = \text{do chain rule again}$$

$$\frac{dy}{dx} = \cos(u) \frac{du}{dx} = \cos(\cos(\tan(x))) \frac{du}{dx}$$

$$y = \cos(\tan(x)), \quad u = \cos(v), \quad v = \tan(x)$$

$$\frac{du}{dx} = \frac{du}{dv} \frac{dv}{dx} \quad \frac{du}{dv} = -\sin(v) \quad \frac{dv}{dx} = \sec^2(x)$$

$$\frac{du}{dx} = -\sin(v) \sec^2(x) = -\sin(\tan(x)) \sec^2(x)$$

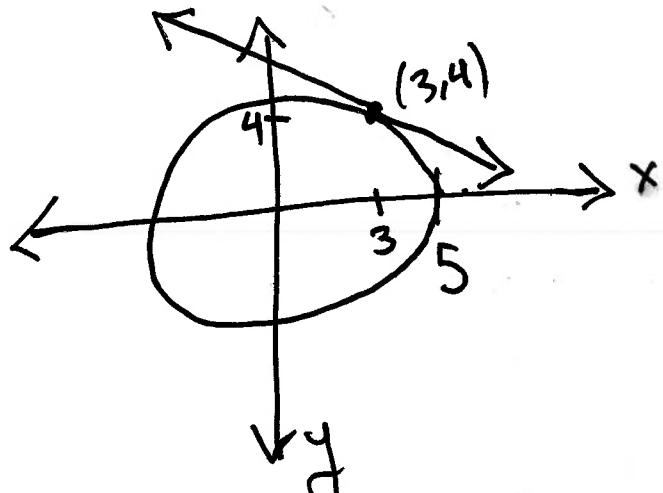
$$\frac{dy}{dx} = -\cos(\cos(\tan(x))) \cdot \sin(\tan(x)) \cdot \sec^2(x)$$

HW

$$\sqrt{x + \sqrt{x + \sqrt{x}}}$$

### 3.5 Implicit Differentiation

$$x^2 + y^2 = 25$$



circle of radius 5  
centered at (0,0)

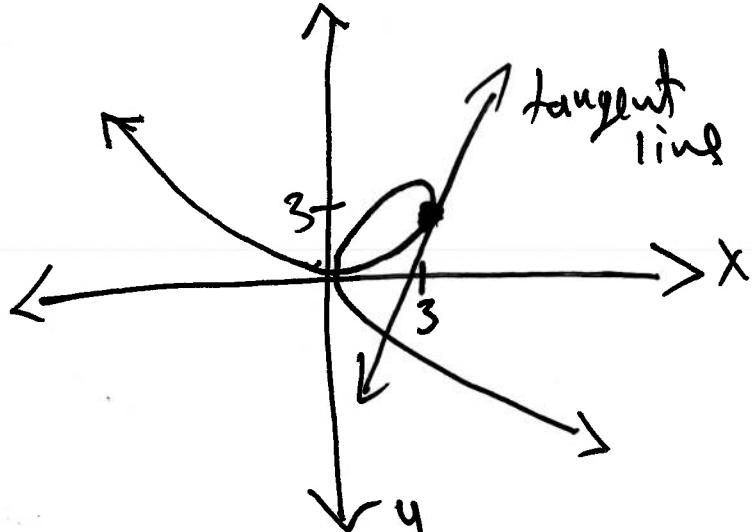
(3, 4) satisfies  $x^2 + y^2 = 25$

$$3^2 + 4^2 = 25$$

$$9 + 16 = 25$$

What's equation  
to  $x^2 + y^2 = 25$   
at (3, 4)?

$$x^3 + y^3 = 6xy$$



folium of Descartes  
(3, 3) satisfies  $x^3 + y^3 = 6xy$

$$3^3 + 3^3 = 6 \cdot 3 \cdot 3$$

$$27 + 27 = 6 \cdot 9$$

$$54 = 54$$

What's equation  
 $x^3 + y^3 = 6xy$  at (3, 3)?

Take derivative

$$2x + 2y = 0$$

Problem: no  $y'$  or  $\frac{dy}{dx}$

Implicit diff. will  
fix the problem.

Solve for  $y$ :

$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \pm \sqrt{25 - x^2}$$

(3,4) is on top circle so use

$$y = \sqrt{25 - x^2}$$

find  $\frac{dy}{dx} = \dots$

Cannot solve for  $y$ ,  
so no way to  
answer the question  
with tools we  
have.

Implicit Differentiation: Apply  $\frac{d}{dx}$  to both sides of equation, thinking of  $y$  as implicitly a function of  $x$ , then solve for  $\frac{dy}{dx}$ . (A  $\frac{dy}{dx}$  form is obtained whenever  $y$  is differentiated by chain rule.)

### Examples

① Find equation of tangent line to  $x^2+y^2=25$  at  $(3,4)$ .

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

Left hand side

$$\begin{aligned} \frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) \\ &= 2x + 2y \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} y^2 &= f(g(x)) \text{ where} \\ f(x) &= x^2, f' = 2x \\ g(x) &= y, g' = \frac{dy}{dx} \\ \text{chain rule} & \end{aligned}$$

$$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$$

Right hand side  $\frac{d}{dx}(2s) = 0$

Put together  $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2s)$

$$2x + 2y \frac{dy}{dx} = 0$$

Solve for  $\frac{dy}{dx}$   $2y \frac{dy}{dx} = -2x$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Slope of tangent line at  $(3, 4)$ : plug  $x=3, y=4$  into  $\frac{dy}{dx}$ .

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$$m = \frac{-3}{4}$$

Equation of tangent line

$$y - 4 = -\frac{3}{4}(x - 3)$$

Q) Find  $y'$  if  $\sin(x+y) = y^2 \cos(x)$ .

$$\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2 \cos(x))$$

LHS

$$\frac{d}{dx}(\sin(x+y)) = \cos(x+y) \cdot \frac{d}{dx}(x+y) \quad \text{by chain rule}$$

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(x) + \frac{d}{dx}(y) = 1 + \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{d}{dx}(\sin(x+y)) = \cos(x+y) \left(1 + \frac{dy}{dx}\right) = \cos(x+y) + \cos(x+y) \frac{dy}{dx}$$

RHS

$$\begin{aligned}\frac{d}{dx}(y^2 \cos(x)) &= \frac{d}{dx}(y^2) \cos(x) + y^2 \cdot \frac{d}{dx}(\cos(x)) \quad \text{product rule} \\ &= 2y \frac{dy}{dx} \cos(x) + y^2 \cdot (-\sin(x)) \\ &= 2y \cos(x) \frac{dy}{dx} - y^2 \sin(x).\end{aligned}$$

Put back together

$$\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2 \cos(x))$$

$$\cos(x+y) + \cos(x+y) \frac{dy}{dx} = 2y \cos(x) \frac{dy}{dx} - y^2 \sin(x)$$

Solve for  $\frac{dy}{dx}$

$$\cos(x+y) + \cos(x+y) \frac{dy}{dx} - 2y \cos(x) \frac{dy}{dx} = -y^2 \sin(x)$$

$$\cos(x+y) \frac{dy}{dx} - 2y \cos(x) \frac{dy}{dx} = -y^2 \sin(x) - \cos(x+y)$$

$$\frac{dy}{dx} (\cos(x+y) - 2y \cos(x)) = -y^2 \sin(x) - \cos(x+y)$$

$$\frac{dy}{dx} = \frac{-y^2 \sin(x) - \cos(x+y)}{\cos(x+y) - 2y \cos(x)}$$

③ Find tangent line to  $x^3 + y^3 = 6xy$  at  $(3,3)$ .

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$

$$\underline{\text{LHS}} \quad \frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$\begin{aligned} \underline{\text{RHS}} \quad \frac{d}{dx}(6xy) &= \frac{d}{dx}(6x)y + 6x \frac{d}{dx}(y) \quad \text{product rule} \\ &= 6y + 6x \cdot 1 \frac{dy}{dx} \end{aligned}$$

Put back together;  $3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$

Solve for  $\frac{dy}{dx}$ :  $3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx} - 3x^2$

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Slope at (3,3): plug in  $x=3, y=3$  into  $\frac{dy}{dx}$ .

$$M = \frac{6 \cdot 3 - 3 \cdot (3)^2}{3 \cdot (3)^2 - 6 \cdot 3} = \frac{18 - 27}{27 - 18} = \frac{-9}{9} = -1$$

Answer 
$$y - 3 = -1 \cdot (x - 3)$$