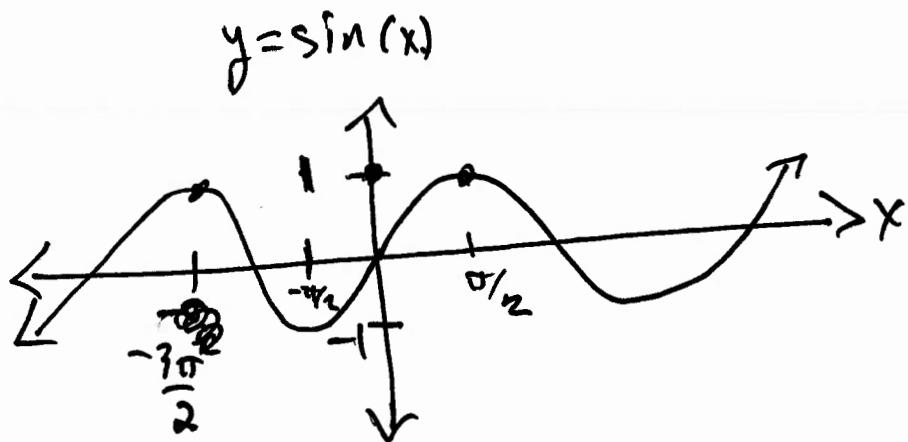
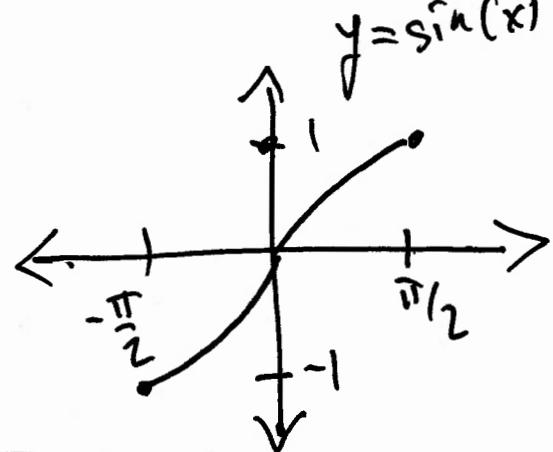


3.6 Inverse Trig Functions and Their Derivatives



Does not pass horizontal line test so no global inverse.

Restrict the domain to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



Does pass horizontal line test.

$\sin^{-1}(x)$ is the inverse function to $\sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$\sin^{-1}(x) = y \iff x = \sin(y)$$

$$-1 \leq x \leq 1$$

domain of $\sin^{-1}(x)$
 $\therefore -1 \leq x \leq 1$

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

aside:

$$a^x = b \iff \log_a b = x$$

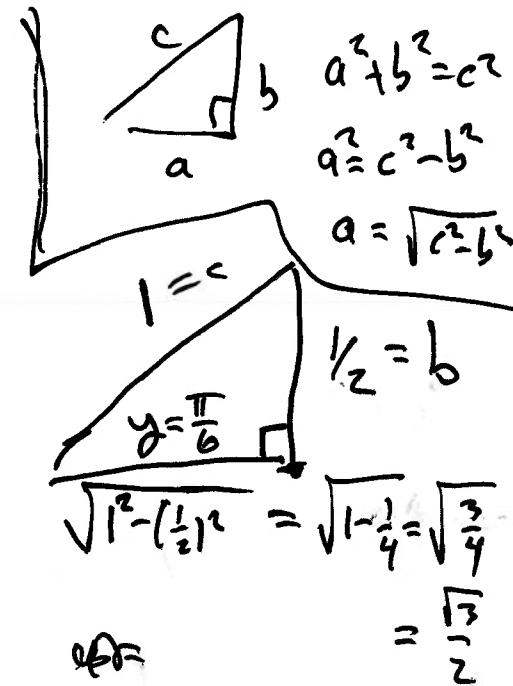
Examples

① Determine $\sin^{-1}\left(\frac{1}{2}\right)$.

$$\sin^{-1}\left(\frac{1}{2}\right) = y \quad \longleftrightarrow \quad \frac{1}{2} = \sin(y)$$

$$\boxed{\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}$$

$$\frac{1}{2} = \sin(y)$$



② $\tan(\sin^{-1}\left(\frac{1}{3}\right))$, $\tan(y)$, $\sin^{-1}\left(\frac{1}{3}\right) = y$

$$\sin^{-1}\left(\frac{1}{3}\right) = y \quad \longleftrightarrow \quad \frac{1}{3} = \sin(y)$$

$$\tan(y) = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{3} \cdot \frac{3}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\boxed{\tan(\sin^{-1}\left(\frac{1}{3}\right)) = \frac{1}{2\sqrt{2}}}$$

$$l = c$$

$$\frac{1}{3} = b$$

$$\frac{2\sqrt{2}}{3} = a = \sqrt{l^2 - \left(\frac{1}{3}\right)^2}$$

$$q = \sqrt{l^2 - \frac{1}{9}}$$

$$q = \sqrt{\frac{9}{9} - \frac{1}{9}} = \sqrt{\frac{8}{9}} = 2\sqrt{2}/3$$

$$\text{Determine } \frac{d}{dx} (\sin^{-1}(x))$$

$$y = \sin^{-1}(x) \longleftrightarrow \sin(y) = x$$

want $\frac{dy}{dx}$

figure out $\frac{dy}{dx}$ using implicit diff.

$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$

$$\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Formula

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \quad \cancel{\text{cancel}}$$

translate back to x's

$$c = \sqrt{c^2 - b^2} = \sqrt{1-x^2}$$

$$a = \sqrt{1-x^2}$$

Formula: $\frac{d}{dx} (\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

Example: $f(x) = \sin^{-1}(\underline{x^2-1})$, determine $f'(x)$.
 inside outside: $\sin^{-1}(\quad)$

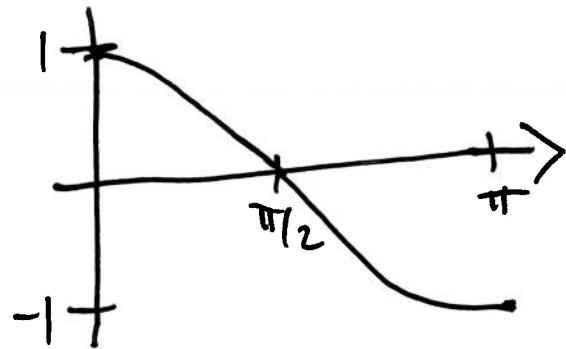
$$y = \sin^{-1}(x^2-1), \quad y = \sin^{-1}(u), \quad u = x^2-1$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}, \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2x = \frac{2x}{\sqrt{1-(x^2-1)^2}}$$

$f'(x) = \frac{2x}{\sqrt{1-(x^2-1)^2}}$

$$y = \cos(x), 0 \leq x \leq \pi$$



$\cos^{-1}(x)$ inverse function to $\cos(x)$

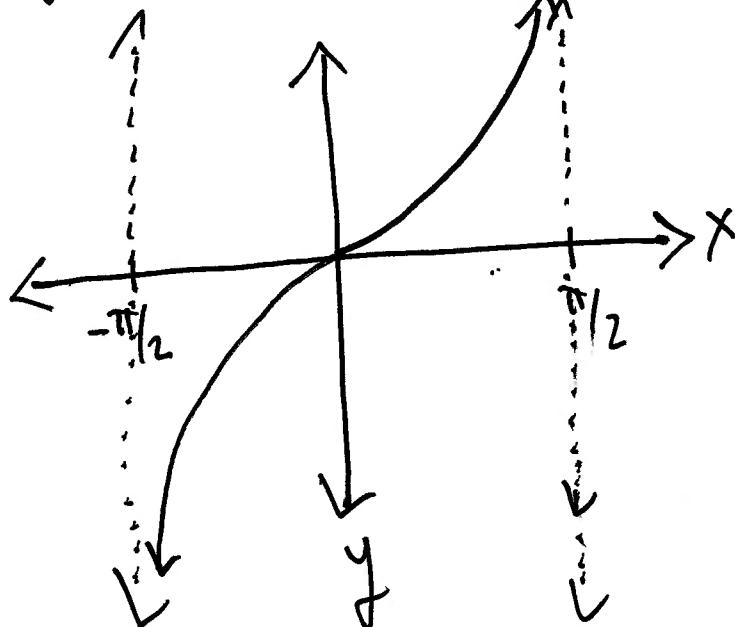
$$\cos^{-1}(x) = y \iff x = \cos(y)$$

$0 \leq y \leq \pi$

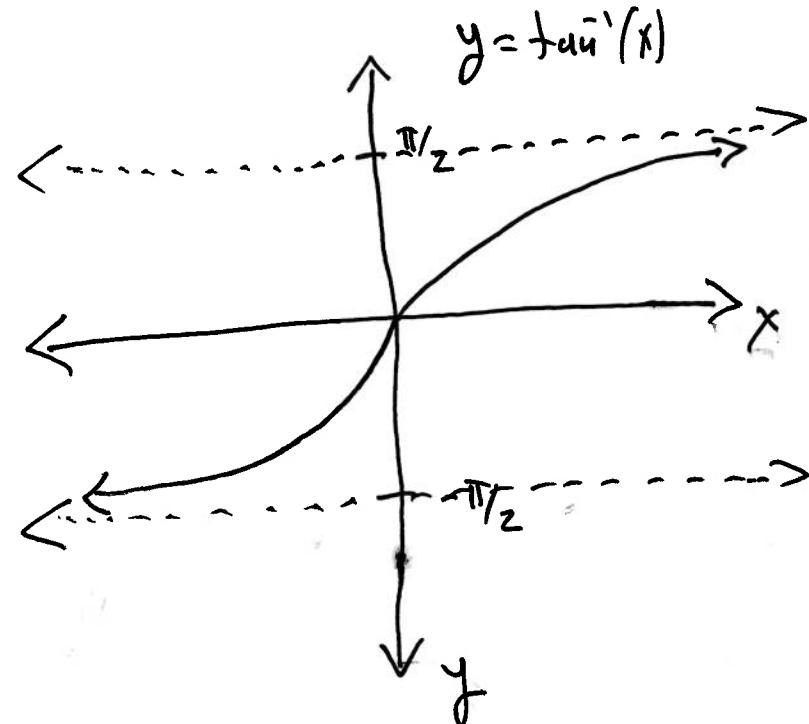
domain of $\cos^{-1}(x)$ is $-1 \leq x \leq 1$

$$\boxed{\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}}$$

$$y = \tan(x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



$$y = \tan'(x)$$



$\tan^{-1}(x)$ inverse function to $\tan(x)$

$$\tan^{-1}(x) = y \longleftrightarrow x = \tan(y), \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

domain of $\tan^{-1}(x)$ is \mathbb{R}

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = \infty$$

$$\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan(x) = -\infty$$

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

Example $\lim_{x \rightarrow 2^+} \tan^{-1}\left(\frac{1}{x-2}\right)$

Plug in: $\tan^{-1}\left(\frac{1}{2-2}\right) = \tan^{-1}\left(\frac{1}{0}\right)$ * more work

know answer is either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = ?, \quad x=3, \quad \frac{1}{3-2} = 1 > 0 \text{ then } \lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$

$$t = \frac{1}{x-2} \quad \text{as } x \rightarrow 2^+, t \rightarrow \infty$$

$$\lim_{x \rightarrow 2^+} \tan^{-1}\left(\frac{1}{x-2}\right) = \lim_{t \rightarrow \infty} \tan^{-1}(t) = \boxed{\frac{\pi}{2}}$$

Determine: $\frac{d}{dx} (\tan^{-1}(x))$

$$y = \tan^{-1}(x) \quad \tan(y) = x$$

$$\frac{d}{dx} (\tan(y)) = \frac{d}{dx} (x)$$

$$\sec^2(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1 + \tan^2(y)}$$

$$s^2 + c^2 = 1$$

$$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$$

$$t^2 + 1 = \sec^2$$

$$\frac{1}{1+x^2}$$

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$$\boxed{\frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1+x^2}}$$

Example $f(x) = \frac{1}{\tan^{-1}(x)}$ determining $f'(x)$

$$y = (\tan^{-1}(x))^{-1}, \quad y = (u)^{-1}, \quad u = \tan^{-1}(x)$$

$$\frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}, \quad \frac{du}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{-1}{u^2} \cdot \frac{1}{1+x^2} = \boxed{\frac{-1}{(\tan^{-1}(x))^2(1+x^2)} = f'(x)}$$

⚠ Notation ⚠

$$\sin^2(x) = (\sin(x))^2$$

$$\sin'(x) \neq (\sin(x))' = \frac{1}{\sin(x)} \quad \text{similarly for } \cos'(x) \text{ and } \tan'(x)$$