Midterm 2
ave. 60%
std. dev. 17%
\[ e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \]

\[ f(x) = \ln(x) \quad [f'(x) = e^x], \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ \frac{1}{x} = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} \]

\[ \frac{\ln(x+h) - \ln(x)}{h} = \frac{\ln \left( \frac{x+h}{x} \right)}{h} = \frac{1}{h} \ln \left( \frac{x+h}{x} \right) = \ln \left( \left( \frac{x+h}{x} \right)^{\frac{1}{h}} \right) \]

\[ \frac{1}{x} = \lim_{h \to 0} \ln \left( \left( 1 + \frac{h}{x} \right)^{\frac{1}{h}} \right) \]

\[ e^x = e^{\lim_{h \to 0} \ln \left( \left( 1 + \frac{h}{x} \right)^{\frac{1}{h}} \right)} = \lim_{h \to 0} e^{\frac{\ln \left( (1 + \frac{h}{x})^{\frac{1}{h}} \right)}{\frac{h}{x}}} = \lim_{h \to 0} (1 + \frac{h}{x})^{\frac{1}{h}} \]
\[ e^{\frac{1}{x}} = \lim_{h \to 0} (1 + \frac{h}{x})^{\frac{1}{h}} \quad n = \frac{1}{h} \quad hu = 1 \quad h = \frac{1}{n} \]

\[ e^{\frac{1}{x}} = \lim_{n \to \infty} (1 + \frac{1}{nx})^{n} \quad \text{as} \quad h \to 0^+, n \to \infty \]

Play in \( x = 1 \) get \( e = \lim_{n \to \infty} (1 + \frac{1}{n})^{n} \)

Substitute \( z = \frac{1}{x} \) \( x = \frac{1}{z} \)

\[ e^{z} = \lim_{n \to \infty} (1 + \frac{z}{n})^{n} \]
4.2 Maximum and Minimum Values

Optimization

- What is the shape of a can that minimizes manufacturing costs?
- What is the radius of a contracted wind pipe that expels air most rapidly during a cough?

Local and absolute mins/maks

\( D = \text{domain of a function } f \)
\( c = \text{number in } D \)

\( f(c) \) is the \underline{absolute maximum} of \( f \) on \( D \) \( \iff \) \( f(c) \geq f(x) \) \( \forall x \in D \)

\( f(c) \) is the \underline{absolute minimum} \( \iff f(c) \leq f(x) \) \( \forall x \in D \)

\( f(c) \) \( \implies \) a \underline{local maximum} \( \iff \) \( f(c) \geq f(x) \) near \( c \)

\( f(c) \) \( \implies \) \underline{local minimum} \( \iff \) \( f(c) \leq f(x) \) near \( c \)
Picture Example

\[ y = f(x) \]

Absolute max

D = [a, b]

- O = local maximums
- \( \square \) = local minimums

More pictures

1. \[ y = x^2, \ D = \mathbb{R} \]

Absolute and local min of 0 at \((0, 0)\)

No absolute or local max.
1.5 \quad y = x^2, \quad D = [-1, 2]

Absolute and local min of 0 at (0,0).
Local max of 1 at (-1,1).
Absolute and local max of 4 at (2,4).

2 \quad y = \sin(x), \quad D = \mathbb{R}

Absolute and local max
y = 1 at \( x = \frac{\pi}{2} + 2\pi k \)
where k is an integer.
Absolute and local min
y = -1 at \( x = \frac{3\pi}{2} + 2\pi k \)
where k is a whole number.
$3)$ \( y = x^3, \quad D = \mathbb{R} \)

No absolute or local maxes or mins.

**Extreme Value Theorem:** If \( f \) is continuous on a closed interval \([a, b]\)
then \( f \) attains an absolute max value \( f(c) \) and an absolute min value \( f(d) \) on some numbers \( c \) and \( d \) in \([a, b]\).
Fermat's Theorem: If \( f \) has a local max or min at \( c \) and \( f'(c) \) is defined, then \( f'(c) = 0 \).

**Def:** A critical number of \( f \) is a number \( c \) in \( D \) where either \( f'(c) = 0 \) or \( f'(c) \) is not defined.

If \( f \) has a local max or min at \( c \), then \( c \) is a critical number for \( f \).
Closed Interval Method

To find the absolute max and min values of a continuous function \( f \) on \( D = [a,b] \):

1. Find critical numbers of \( f \) in \((a,b)\)
2. Find the \( y \)-values of \( f \) at critical numbers.
3. Find the \( y \)-values of \( f \) at the end points of interval \( (x=a, x=b) \).

3. The largest of the \( y \)-values from steps 1 and 2 is the absolute maximum. The smallest is the absolute minimum.

Examples

1. \( f(x) = 2x^3 - 3x^2 - 12x + 1 \), \( D = [-2,3] \)
   Find absolute max and min of \( f \) on \( D \).
Step 0

\[ f'(x) = 6x^2 - 6x - 12 \]

0 = 6x^2 - 6x - 12
0 = x^2 - x - 2
0 = (x - 2)(x + 1)

\[ x = 2 \text{ or } x = -1 \]

Critical numbers are \( x = 2, x = -1 \)

Step 1

\[ f(2) = ? , \quad f(-1) = ? \]

\[ f(2) = 2 \cdot (2)^3 - 3 \cdot (2)^2 - 12 \cdot 2 + 1 \]

= 2 \cdot 8 - 3 \cdot 4 - 24 + 1
= 16 - 12 - 23 = 4 - 23 = -19

\[ f(-1) = 2 \cdot (-1)^3 - 3 \cdot (-1)^2 - 12 \cdot (-1) + 1 \]

= -2 - 3 + 12 + 1 = -5 + 13 = 8

\[ f(2) = -19 \]

\[ f(-1) = 8 \]

Step 2

\[ a = -2, \quad b = 3, \quad f(-2) = ?, \quad f(3) = ? \]
\[ f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1 \]
\[ = 2(-8) - 3(4) + 24 + 1 \]
\[ = -16 - 12 + 25 = -28 + 25 = -3 \]
\[ f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 1 \]
\[ = 2(27) - 3(9) - 36 + 1 \]
\[ = 54 - 27 - 35 = 27 - 35 = -8 \]

**Step 3**

- \[ f(2) = -19, \]  
- \[ f(-1) = 8, \]  
- \[ f(-2) = -3, \]  
- \[ f(3) = -8 \]

\[ f \] has an absolute max of 8 at \( x = -1 \)

\[ f \] has an absolute min of 19 at \( x = 2 \)