

Closed Interval Method

- find absolute max/min of f on $[a,b]$
0. find critical numbers ($f' = 0$)
 1. plug into f critical numbers
 2. plug into f $x=a, x=b$
 3. Largest value = absolute max
smallest value = absolute min.

Example

$$f(x) = x - 2\sin(x), [0, 2\pi]$$

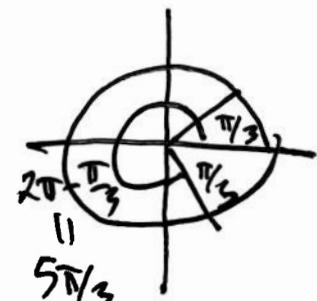
$$0. f'(x) = 1 - 2\cos(x)$$

$$0 = 1 - 2\cos(x)$$

$$\cos(x) = \frac{1}{2}$$

$$x = \frac{\pi}{3}, x = \frac{5\pi}{3}$$

critical numbers



$$1. f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{2\sqrt{3}}{2} = \frac{\pi}{3} - \sqrt{3}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \frac{2\sqrt{3}}{2} = \frac{5\pi}{3} + \sqrt{3}$$

$$2. f(0) = 0 - 2\sin(0) = 0$$

$$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi$$

$$3. f(0) = 0, f(2\pi) = 2\pi, f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \approx -0.68 \leftarrow \begin{matrix} \text{abs.} \\ \text{min} \end{matrix}$$

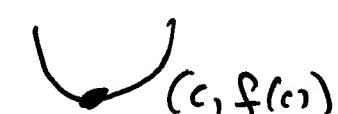
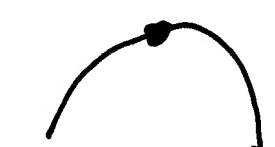
$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97 \leftarrow \begin{matrix} \text{abs. max} \\ \text{max} \end{matrix}$$

4.3 Derivatives and Shapes of Curves

Rules $I = \text{interval}^{\text{open}} (I = (a, b))$, $f(x)$ - function.

- (a) If $f'(x) > 0$ for all x in I , then f is increasing on I . ↗
- (b) If $f'(x) < 0$ for all x in I , then f is decreasing on I . ↘
- (c) If $f''(x) > 0$ for all x in I , then f is concave up on I . ⌟
- (d) If $f''(x) < 0$ for all x in I , then f is concave down on I . ⌠

2nd Derivative Test

- (e) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x=c$.
 ($f(c)$ is the local min.) 
- (f) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x=c$.
 ($f(c)$ is the local max.) 

Points of Inflection

- (g) A point of inflection for $f(x)$ is a point where the concavity of f changes. (If c is a p.o.i., then $f''(c)=0$.)

Example Determine (a)-(g) for $f(x) = x^4 - 4x^3$ and use this to sketch the graph of f .

(a), (b)

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x-3)$$

$$0 = 4x^2(x-3)$$

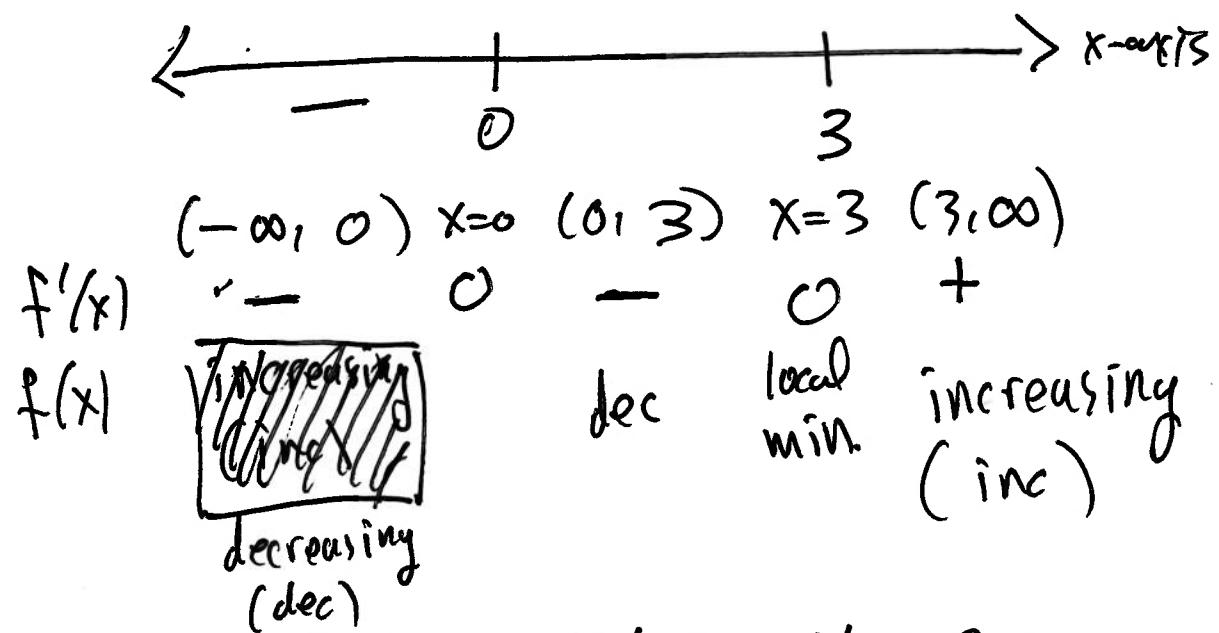
$$x=0 \text{ or } x=3$$

Test points

$$x=-1 \text{ in } (-\infty, 0),$$

$$x=1 \text{ in } (0, 3),$$

$$x=4 \text{ in } (3, \infty),$$



$$f'(-1) = 4(-1)^2(-1-3) = 4(-4) = -16 < 0$$

$$f'(1) = 4(1)^2(1-3) = 4(-2) = \boxed{-8} < 0$$

$$f'(4) = 4(4)^2(4-3) = 4^3 > 0$$

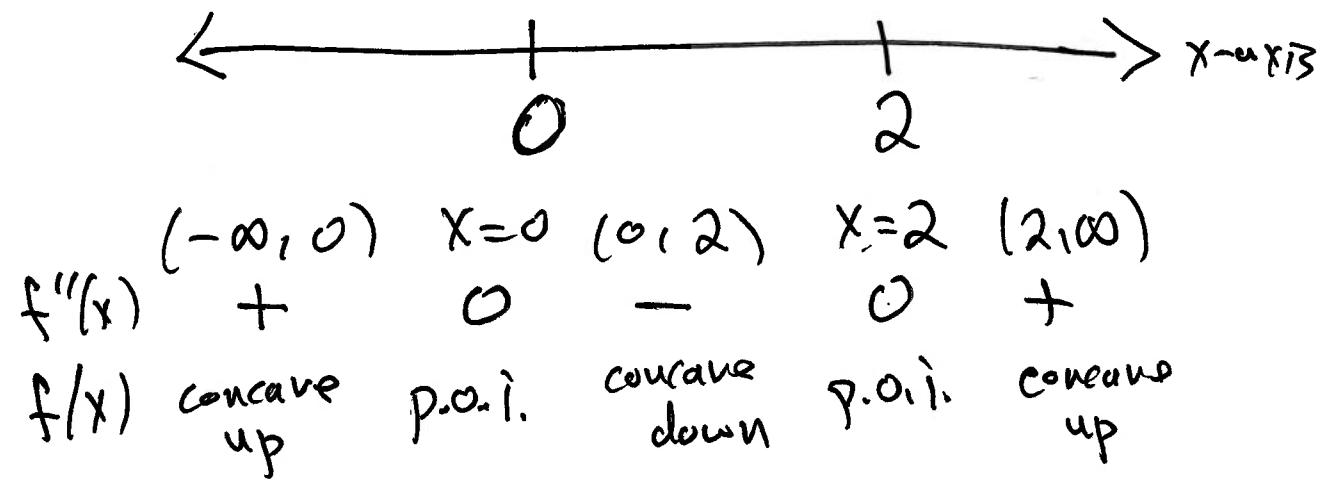
(c), (d)

$$f''(x) = 12x^2 - 24x$$

$$f''(x) = 12x(x-2)$$

$$0 = 12x(x-2)$$

$$x=0 \text{ or } x=2$$



Test Points

$$x=-1 \text{ is in } (-\infty, 0),$$

$$f''(-1) = 12(-1)(-1-2) = -12(-3) = 36 > 0$$

$$x=1 \text{ is in } (0, 2),$$

$$f''(1) = 12(1)(1-2) = 12(-1) = -12 < 0$$

$$x=3 \text{ is in } (2, \infty),$$

$$f''(3) = 12 \cdot 3(3-2) = 36 \cdot 1 = 36 > 0$$

graph $f(x) = x^4 - 4x^3 = x^3(x-4)$

important points $x=0, x=2, x=3$

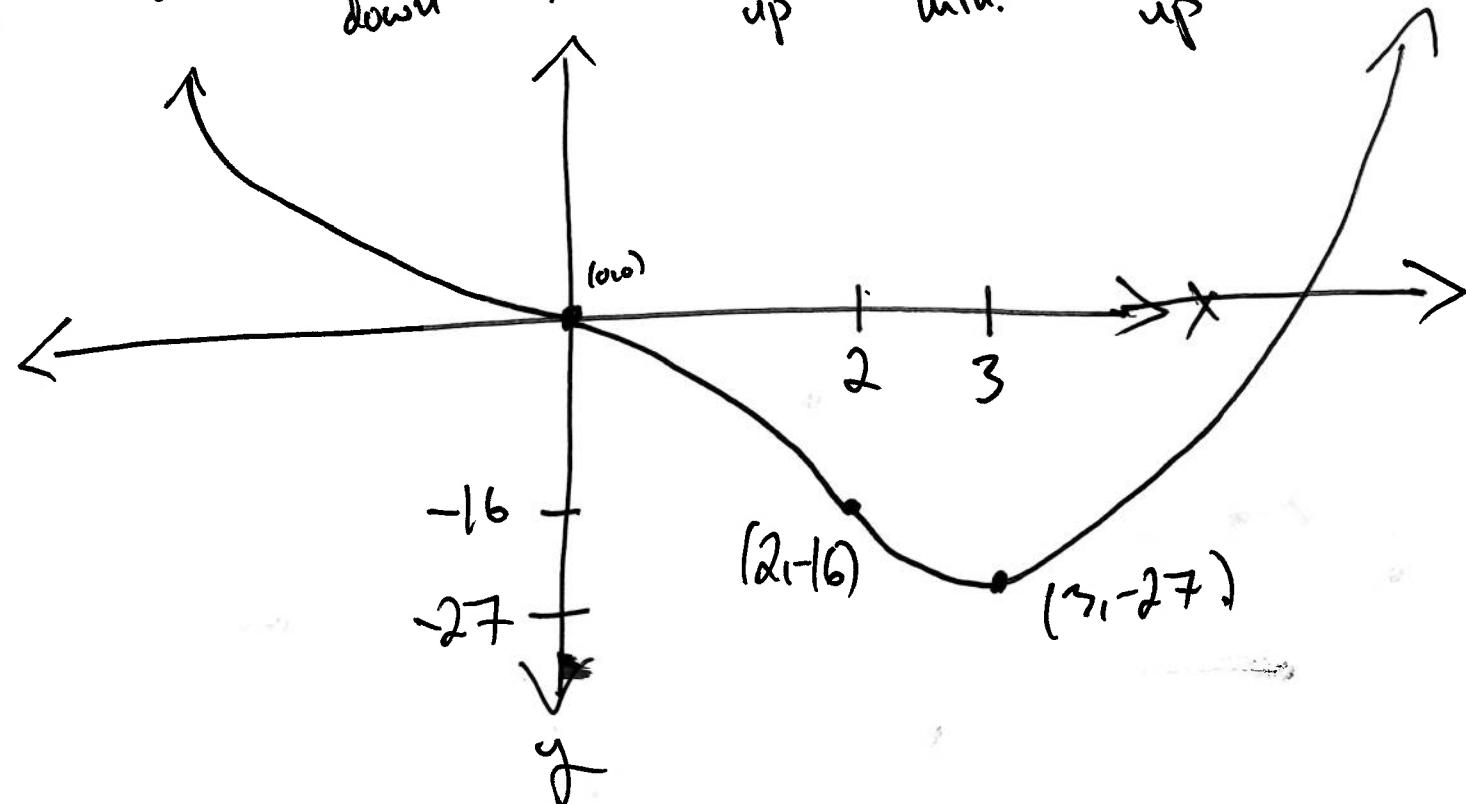
$$f(0) = 0^4 - 4 \cdot 0^3 = 0$$

$$f(2) = 2^3(2-4) = 8(-2) = -16$$

$$f(3) = 3^3(3-4) = 27(-1) = -27$$

$(-\infty, 0) \ x=0 \ (0, 2) \ x=2 \ (2, 3) \ x=3 \ (3, \infty)$

$f'(x)$	-	0	-	-	-	0	+
$f''(x)$	+	0	-	0	+	+	+
$f(x)$	dec. concave up	p.o.i. dec	dec. concave down	dec. p.o.i.	dec concave up	local min.	inc. concave up



Example $f(x) = e^{-(x-1)^2} = e^{-(x^2-2x+1)} = e^{-x^2+2x-1}$

$$f(x) = e^{-x^2+2x-1}$$

$$y = e^{-x^2+2x-1}$$

$$y = e^u$$

$$u = -x^2+2x-1$$

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dx} = -2x+2$$

$$\frac{dy}{dx} = e^u (-2x+2) = (-2x+2) e^{-x^2+2x-1}$$

$$f'(x) = (-2x+2) e^{-x^2+2x-1}$$

$$f''(x) = -2e^{-x^2+2x-1} + (-2x+2)(-2x+2)e^{-x^2+2x-1}$$

$$f''(x) = -2e^{-x^2+2x-1} + (-2x+2)^2 e^{-x^2+2x-1}$$

$$f''(x) = (-2 + (-2x+2)^2) e^{-x^2+2x-1}$$

$$f''(x) = (-2 + 4x^2 + 2 \cdot (-2x) \cdot 2 + 4) e^{-x^2+2x-1}$$

$$f''(x) = (4x^2 - 8x + 2) e^{-x^2+2x-1}$$

$$e^{-x^2+2x-1} > 0$$

for all x