

## 1.6 Inverse Functions and Logarithms

EX

$$f(t) = 4t + 100 \quad \text{population of bacteria in an experiment}$$

at time  $t$

Question: How long does it take until there is  $N$  bacteria?

Answer:  $N = 4t + 100$

$$N - 100 = 4t$$

$$\frac{N - 100}{4} = t$$

$$\frac{1}{4}N - 25 = t$$

New function:

$$g(N) = \frac{1}{4}N - 25$$

$g(N)$  tells you the amount  
of time it takes for  
there to be  $N$  bacteria

$f(x) = 4x + 100$  and  $g(x) = \frac{1}{4}x - 25$  are inverse functions

to each other ( $g = f^{-1}$ ,  $f = g^{-1}$ ).

A function  $f$  is called one-to-one if it never takes on the same value twice; that is if  $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$

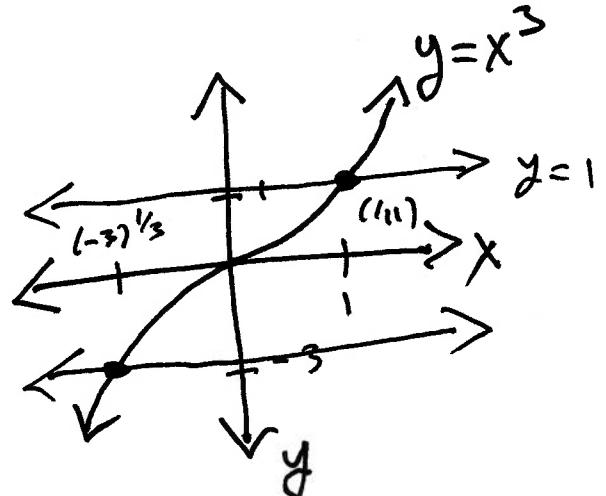
Horizontal line test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

### Exs

①  $f(x) = x^3$

If  $x_1 \neq x_2$ , then  $f(x_1) = x_1^3 \neq x_2^3 = f(x_2)$  so  $f$  is one-to-one. Graph

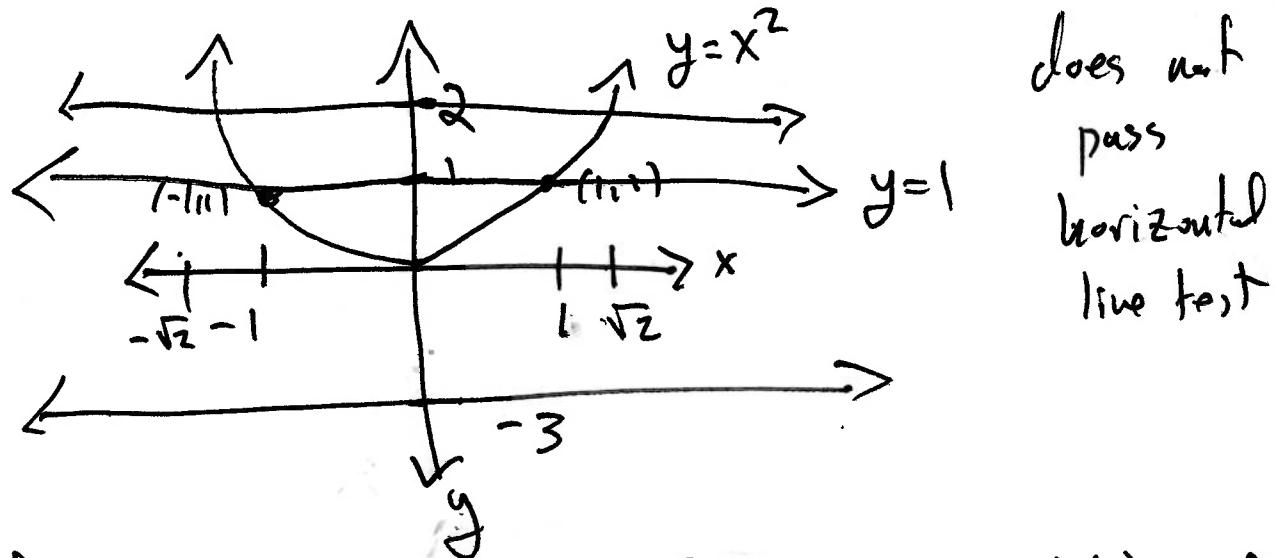
$y = x^3$  passes horizontal line test



$$\textcircled{2} \quad f(x) = x^2$$

We have that  $1 \neq -1$  but  $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$ ,  
so  $f$  is not one-to-one.

Graph



A function  $f$  has an inverse (or is invertible) if and only if  $f$  is one-to-one. The inverse of  $f$  is denoted  $f^{-1}$ , and is defined as

$$f^{-1}(y) = x \quad \text{where } x \text{ is such that } f(x) = y.$$

The domain of  $f^{-1}$  is the range of  $f$ .

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Ex: If  $f(1)=5$  and  $f(3)=7$ , what is  $f'(5)$  and  $f'(7)$ ?

$f'(5) = x$  where  $f(x)=5$ , so  $x=1$  so  $f'(5)=1$

$f'(7) = x$  where  $f(x)=7$ , so  $x=3$ , so  $f'(7)=3$

$\triangle$  Warning:  $f'(x) \neq \frac{1}{f(x)}$   $\triangle$  (Notational Warning)



$$[f(x)]^{-1} = \frac{1}{f(x)}$$

$$\sin^2(x) = (\sin(x))^2 \quad \sin^{-1}(x) \neq (\sin(x))^{-1}$$

$\nwarrow$  means inverse function

### Important Property of Inverse Functions

$$f \circ f^{-1}(x) = x \quad \text{for all } x \text{ in range of } f$$

$$f^{-1} \circ f(x) = x \quad \text{for all } x \text{ in domain of } f$$

## Calculating the inverse of $f(x)$

1. Set  $y = f(x)$

2. Solve for  $x$  in terms of  $y$ ,  $x = g(y)$

3. Switch  $x$  and  $y$  so  $y = g(x)$ . Then  $f^{-1}(x) = g(x)$ .

Examples Find inverse of  $f$

①  $f(x) = 1 + \sqrt{2+3x}$ ,  $x \geq -\frac{2}{3}$

$$y = 1 + \sqrt{2+3x}$$

$$y-1 = \sqrt{2+3x}$$

$$(y-1)^2 = 2+3x$$

$$(y-1)^2 - 2 = 3x$$

$$\frac{(y-1)^2 - 2}{3} = x$$

switch  
 $x$  and  $y$

$$y = \frac{(x-1)^2 - 2}{3}$$

$$\boxed{f^{-1}(x) = \frac{(x-1)^2 - 2}{3}}$$

②  $f(x) = \frac{3x+1}{x-3}$

$$y = \frac{3x+1}{x-3}$$

$$y = 3 + \frac{10}{x-3}$$

$$y-3 = \frac{10}{x-3}$$

$$(y-3)(x-3) = 10$$

$$x-3 = \frac{10}{y-3}$$

$$x = \frac{10}{y-3} + 3$$

$$\begin{array}{r} 3 + \frac{10}{x-3} \\ x-3 \overline{)3x+1} \\ \underline{3x-9} \\ \hline 10 \end{array}$$

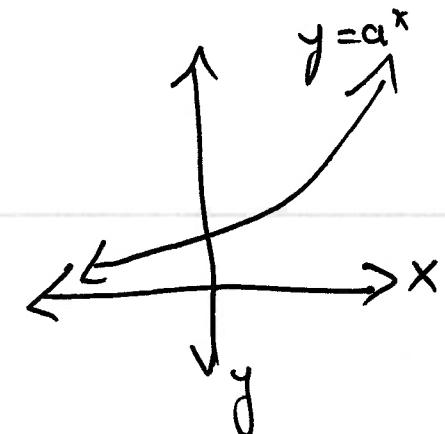
$$\frac{3x+1}{x-3} = 3 + \frac{10}{x-3}$$

*switch  
x and y*  $\rightarrow y = \frac{10}{x-3} + 3$

$$\boxed{f^{-1}(x) = \frac{10}{x-3} + 3}$$

## Logarithmic Functions

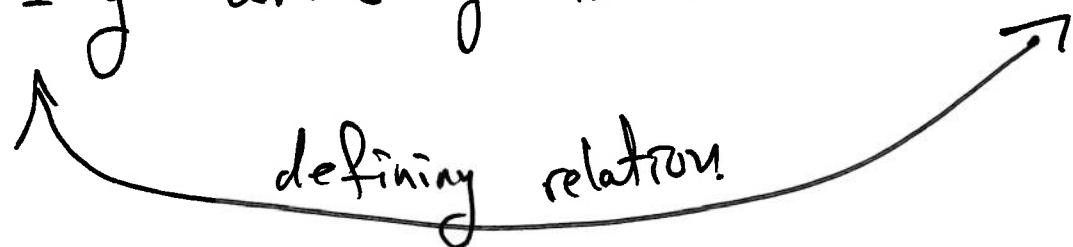
$f(x) = a^x$  passes horizontal line test



$\log_a x$  = the inverse function of  $f(x) = a^x$ .

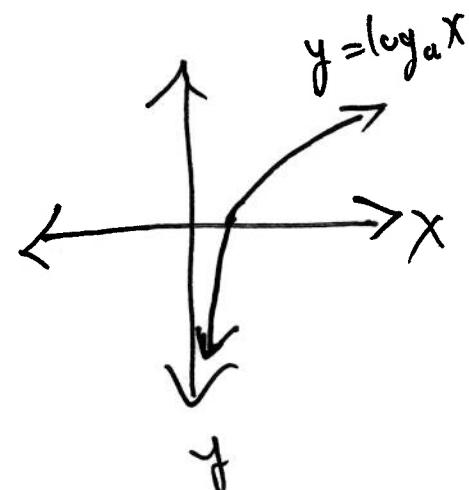
$\log_a x$  is defined by the rule

$\log_a x = y$  where  $y$  is such that  $a^y = x$



## Properties of $\log_a (x)$

1.  $(0, \infty) = \text{domain}, \mathbb{R} = \text{range}$



$$2. \log_a(a^x) = x \text{ for any } x \in \mathbb{R}$$

$$a^{\log_a(x)} = x \text{ for any } x \in (0, \infty)$$

$$3. \log_a(xy) = \log_a(x) + \log_a(y)$$

$$4. \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$5. \log_a(x^r) = r \log_a(x)$$

Natural logarithm

$e = \text{Euler's number} = 2.71828\dots$  very important number.

$$\ln x = \log_e x$$

↑  
called the natural logarithm

(sometimes  
 $\log x = \log_e x$ )

Ex

Find inverse of  $f(x) = e^{2x-1}$

$$y = e^{2x-1}$$

Take  $\ln(\cdot)$  of both sides

$$\ln(y) = \ln(e^{2x-1})$$

By property 5

$$\ln(y) = (2x-1)\ln(e)$$

$$\ln(y) = 2x-1$$

$$1 + \ln(y) = 2x$$

$$\frac{1 + \ln(y)}{2} = x$$

By property 2

$$\ln(e^t) = t$$

$$f^{-1}(x) = \frac{1 + \ln(x)}{2}$$