

2.3 Calculating Limits Using Limit Laws

Evaluating limits using algebra

Examples

① Polynomial Factoring: Determine $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$

Previously, we looked at a table or said $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \frac{1}{2}$.

"Plug in $x=1$ " $\frac{1-1}{1^2-1} = \frac{0}{0}$ indeterminate form
this makes no sense
need to do more work

Factor

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$$

If $x \neq 1$, then $\frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$.

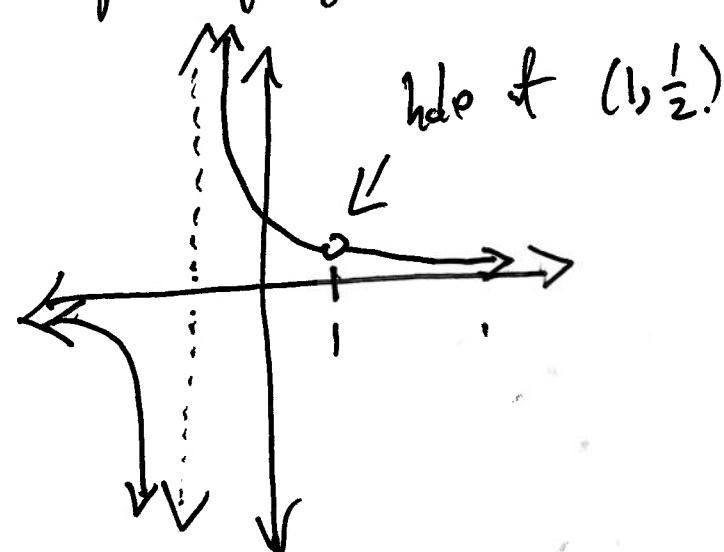
Then $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{1}{x+1}$. ~~•~~

Now we can plug in $x=1$:

$$\lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

Graph of $f(x) = \frac{x-1}{x^2-1}$ is the graph of $g(x) = \frac{1}{x+1}$

everywhere except at $x=1$.



② Expanding, Simplifying and Cancelling: Determine $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$

"Plug in $h=0$ ": $\frac{(3+0)^2 - 9}{0} = \frac{9-9}{0} = \frac{0}{0}$, so need to do more work.

Expand, simplify, and cancel:

$$\begin{aligned}\frac{(3+h)^2 - 9}{h} &= \frac{9 + 2 \cdot 3h + h^2 - 9}{h} \\ &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{6h + h^2}{h} = \frac{h(6+h)}{h}.\end{aligned}$$

If $h \neq 0$, then $\frac{h(6+h)}{h} = 6+h$.

Then $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} 6+h = 6+0 = \boxed{6}$.

③ Rationalizing: Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2}$

"Plug in $t=0$ ": $\frac{\sqrt{0^2+9} - 3}{0^2} = \frac{\sqrt{9} - 3}{0} = \frac{3-3}{0} = \frac{0}{0}$,

If $t \neq 0$, then

so must do more work.

Rationalize: $\frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} =$

$$= \frac{-\sqrt{t^2+9} \sqrt{t^2+9} - 3 \sqrt{t^2+9} + 3 \sqrt{t^2+9} + 9}{t^2(-\sqrt{t^2+9} - 3)} =$$

$$= \frac{-(t^2+9) + 9}{t^2(-\sqrt{t^2+9} - 3)} = \frac{-t^2 - 9 + 9}{t^2(-\sqrt{t^2+9} - 3)}$$

$$= \frac{-t^2}{t^2(-\sqrt{t^2+9} - 3)}$$

$$= \frac{-1}{-\sqrt{t^2+9} - 3}.$$

We just showed that if $t \neq 0$, then

$$\frac{\sqrt{t^2+9} - 3}{t^2} = \frac{-1}{-\sqrt{t^2+9} - 3}.$$

Then

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{-1}{-\sqrt{t^2+9} - 3} = \frac{-1}{-\sqrt{0^2+9} - 3} \\ &= \frac{-1}{-\sqrt{9} - 3} \end{aligned}$$

$$= \frac{-1}{-3-3} = \frac{-1}{-6} = \boxed{\frac{1}{6}}$$

Squeeze Theorem

(want to calculate $\lim_{x \rightarrow a} \underline{\overbrace{g(x)}}$)

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a), then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \leq \lim_{x \rightarrow a} h(x).$$

In particular if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then

$$\lim_{x \rightarrow a} g(x) = L.$$

Example Determine $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$: (Remember that

$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exist because of oscillation.)

Want to pick $f(x), h(x)$ such that

$$f(x) \leq x^2 \sin\left(\frac{1}{x}\right) \leq h(x)$$

and $f(0) = h(0)$.

BB We know

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for all $x \neq 0$. Multiplying by x^2 on all sides gives

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

We take $f(x) = -x^2$ and $h(x) = x^2$. Then by the squeeze theorem

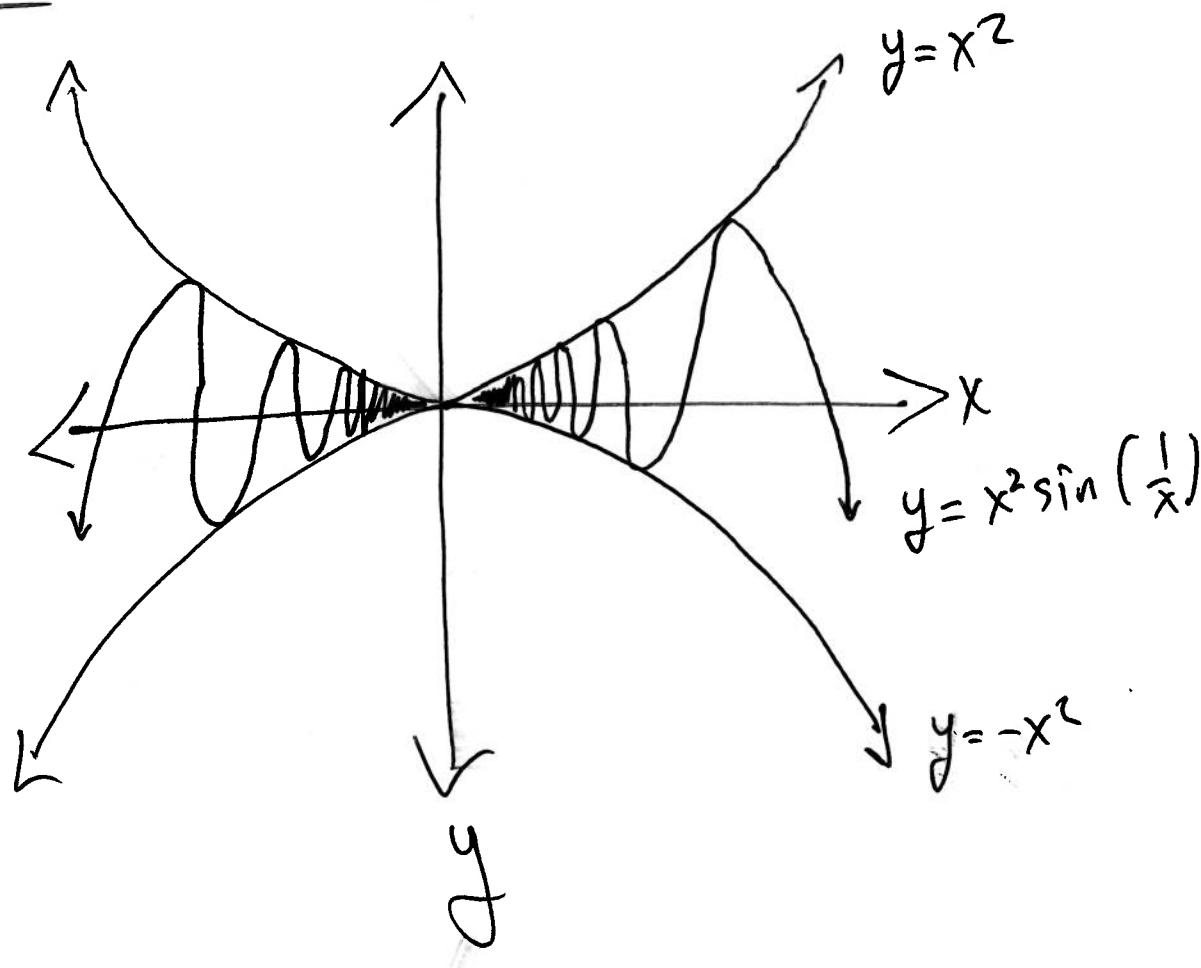
$$\lim_{x \rightarrow 0} -x^2 \leq \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0} x^2$$

$$0 \leq \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

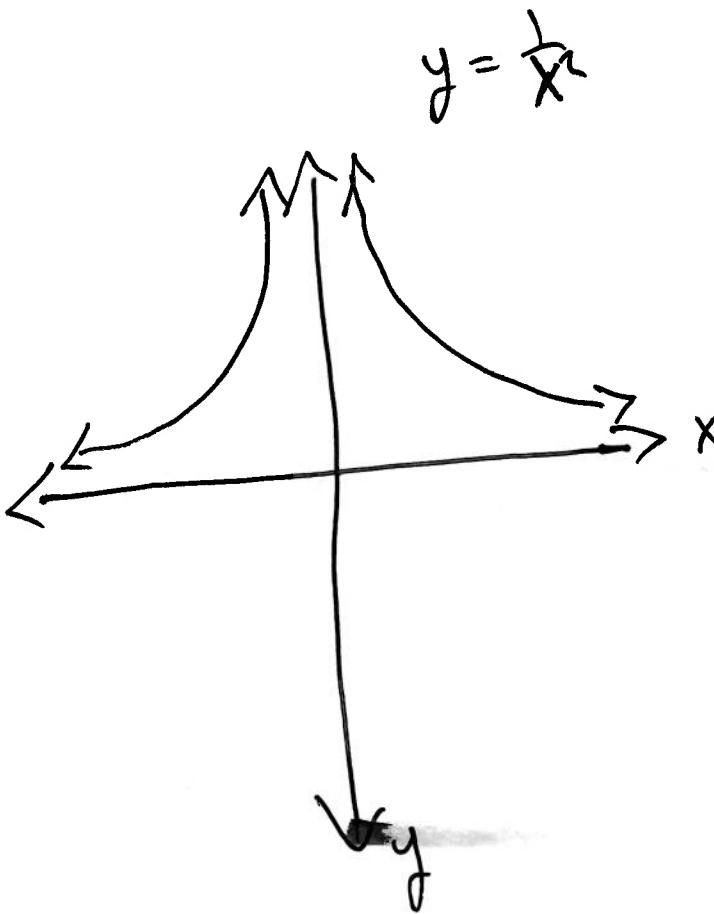
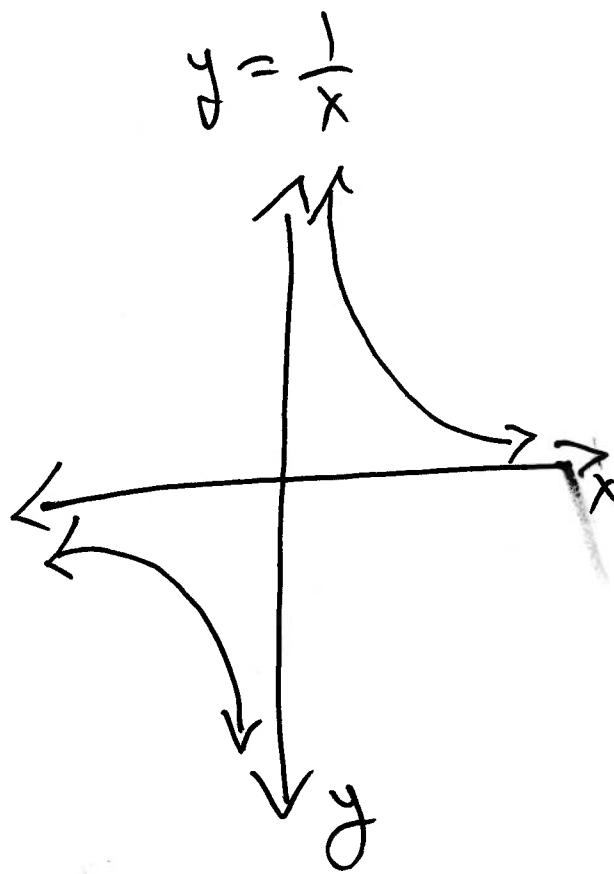
(because $\lim_{x \rightarrow 0} -x^2 = 0$ and $\lim_{x \rightarrow 0} x^2 = 0$)

Then $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Picture



2.5 Limits Involving Infinity



$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ does not exist}$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \infty$$

1

Def: $\lim_{x \rightarrow a} f(x) = \infty$ means the values of $f(x)$ can be made arbitrarily large (as large as we want) by taking x sufficiently close a (on either side of a) but not equal to a .

$\lim_{x \rightarrow a} f(x) = -\infty$ means same with arbitrarily large replaced with arbitrarily negatively large.

