Symmetry Group Problems

1. Let G be the symmetry group of a tetrahedral centered at the origin in $\mathbb{R}^3$. Let E be the set of edges of the tetrahedral. Show that G acts on E, giving a homomorphism from G to $\mathrm{Sym}(6)$ (because there are 6 edges of a tetrahedral). Show that this action is transitive.

2. Let G be the symmetry group of a tetrahedral as in problem one. Now let V be the set of vertices of the tetrahedral. Show that G acts on V, giving a homomorphism from G to $\mathrm{Sym}(4)$. Show that G is isomorphic to $\mathrm{Sym}(4)$ by showing that the homomorphism induced by the action is an isomorphism.

3. Let G be the symmetry group of a tetrahedral as in problems 1 and 2. Let

$$\varphi : G \longrightarrow \mathrm{Sym}(4)$$

be the isomorphism induced by the action of G on the vertices of the tetrahedral. By definition, G is a subset of $\mathrm{GL}_3(\mathbb{R})$. Let $\iota : G \to \mathrm{GL}_3(\mathbb{R})$ be the inclusion homomorphism. Show that the composition homomorphism

$$\det \circ \iota \circ \varphi^{-1} : \mathrm{Sym}(4) \xrightarrow{\varphi^{-1}} G \xrightarrow{\iota} \mathrm{GL}_3(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times$$

is the sign homomorphism $\mathrm{sgn} : \mathrm{Sym}(4) \to \{\pm 1\}$.

4. Let G be the group of rotations of a cube centered as the origin in $\mathbb{R}^3$. Show that $G \cong \mathrm{Sym}(4)$.

5. Observe that $D_6 \subset D_{12}$ because we can choose 3 vertices of a hexagon that are the vertices of an equilateral triangle. Then the action of D_{12} on the 6 vertices of the hexagon induces a homomorphism

$$D_6 \longrightarrow D_{12} \longrightarrow \mathrm{Sym}(6)$$

Show that this action of D_6 on the 6 vertices of the hexagon is not transitive.