Normal subgps and factor groups

Motivation: Let G be a gp and $H \subseteq G$ a subgp of G. We have the left cosets of H in G:

$$G/H = \{aH : a \in G\}$$

Question: Can we define a binary operation on G/H by the rule

$$aH \cdot cH = acH \,?$$

One necessary condition for this binary operation to be well defined is that if $aH = bH$, then $acH = bcH$.

We know

$$aH = bH \quad \text{iff} \quad a^{-1}b \in H$$

$$acH = bcH \quad \text{iff} \quad (ac)^{-1}bc = c^{-1}a^{-1}bc \in H$$

If H has the property that $\forall g \in G, h \in H$, $ghg^{-1} \in H$, then if $aH = bH$, then $acH = bcH$ because

$$c^{-1}\underbrace{a^{-1}b}_{\in H}c \in H \quad \text{since } a^{-1}b \in H \quad (\text{take } g = c^{-1}, h = a^{-1}b).$$

Def: Let $H \subseteq G$ be a subgp of a gp G. For $g \in G$, define the subset of G

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

We say that H is a <u>normal</u> subgp of G if $\forall g \in G$

$$gHg^{-1} = H$$

<u>or</u> equivalently if $\forall g \in G, h \in H$, $ghg^{-1} \in H$. We denote H being a normal subgroup of G by $H \trianglelefteq G$, or $H \triangleleft G$.

__Remarks__: 1. Observe that this is __not__ saying that given $g \in G$, $h \in H$, $ghg^{-1} = h$. It is merely saying that $ghg^{-1} \in H$.

2. If G is abelian then every subgp is normal since $\forall g, h \in G$, $ghg^{-1} = gg^{-1}h = h$.

3. Let $G = \mathrm{Sym}(3)$, $H = \{1, (12)\}$. Then $(13)(12)(13) = (23) \notin H$, so H is not normal. Therefore not all subgps of a group are normal.

__Prop__: Let $H \subseteq G$ be a normal subgp. Then the binary operation

$$G/H \times G/H \longrightarrow G/H$$
$$(xH, yH) \longmapsto (xyH)$$

is well-defined.

__Proof__: We need to show that if $aH = xH$ and $bH = yH$, then $abH = xyH$. Assume $aH = xH$, so $a^{-1}x \in H$

$bH = yH$, so $b^{-1}y \in H$.

We need to show that $(ab)^{-1}xy \in H$. We have

$$(ab)^{-1}xy = b^{-1}a^{-1}xy$$
$$= b^{-1}yy^{-1}a^{-1}xy$$
$$= b^{-1}y(y^{-1}a^{-1}xy)$$

H is a normal subgp implies $y^{-1}a^{-1}xy \in H$ since $a^{-1}x \in H$. Therefore $(ab)^{-1}xy \in H$ since $(ab)^{-1}xy = b^{-1}y(y^{-1}a^{-1}xy)$ and $b^{-1}y, y^{-1}a^{-1}xy \in H$. □

Prop: Let $H \subseteq G$ be a normal subgp of a gp. Then G/H is a gp under the above binary operation. We call this group the <u>quotient group</u> of G by H.

<u>proof</u>: 1. (identity) We claim that $H \in G/H$ is ~~H~~ an identity element: $\forall g \in G$ we have

$$H \cdot gH = 1 \cdot gH = gH = g \cdot 1H = gH \cdot H$$

2. (associativity) Let $xH, yH, zH \in G/H$. Then

$$\begin{aligned}(xH \cdot yH) \cdot zH &= xyH \cdot zH \\ &= xyzH \quad \text{because } G \text{ is associative} \\ &= xH\, yzH \\ &= xH \cdot (yH \cdot zH)\end{aligned}$$

3. (inverses) We claim that $x^{-1}H = (xH)^{-1}$ for all $xH \in G/H$.

$$x^{-1}H\, xH = x^{-1}xH = 1 \cdot H = H = xx^{-1}H = xH\, x^{-1}H. \quad \square$$

Prop: The map $\pi : G \to G/H$, $x \mapsto xH$ is a gp homomorphism if H is a normal subgp of G.

And $\ker(\pi) = H$.

<u>Proof</u>: Let $x, y \in G$. Then $\pi(xy) = xyH = xH\,yH = \pi(x)\pi(y)$, so π is a gp hom. $\ker(\pi) = \{x \in G : \pi(x) = H\}$. Now $\pi(x) = xH = H$ iff $x \in H$, so $\ker(\pi) = H$. $\square$

<u>Remark</u>: The previous prop shows that any normal subgp can be realized as the kernel of a gp homomorphism.

Examples (of quotient groups when G is abelian

1. $n\mathbb{Z} \subseteq \mathbb{Z}$ normal since $\mathbb{Z}$ is abelian

   $\mathbb{Z}/n\mathbb{Z}$ - quotient gp

2. $G = \mathbb{Z}\times\mathbb{Z}$, $H = \{(x,0) : x \in \mathbb{Z}\}$, $H \subseteq \mathbb{Z}\times\mathbb{Z}$ subgp

   claim: $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}/H$ is a gp isomorphism.

   $$\varphi(y) = (0,y) + H$$

   proof: φ is a gp hom: Let $y, z \in \mathbb{Z}$, then

   $$\varphi(y+z) = (0,y+z) + H = (0,y) + (0,z) + H$$
   $$= (0,y) + H + (0,z) + H = \varphi(y) + \varphi(z)$$

   Injective: Say $\varphi(y) = \varphi(z)$. This means $(0,y) + H = (0,z) + H$.

   $(0,y) + H = (0,z) + H$ iff $(0,y) - (0,z) = (0, y-z) \in H$

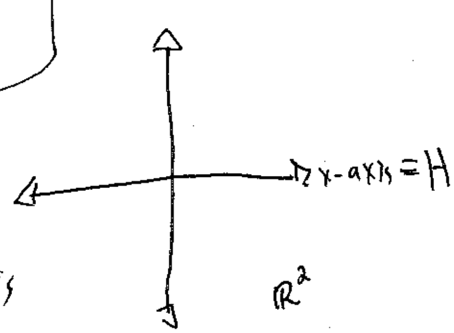
   iff $y - z = 0$

   iff $y = z$

   Surjective: Let $(x,y) + H \in \mathbb{Z}\times\mathbb{Z}/H$. Then $(x,y) + H = (0,y) + H$ because $(x,y) - (0,y) = (x,0) \in H$. Then

   $$\varphi(y) = (0,y) + H = (x,y) + H. \quad \square$$

   Hence $\mathbb{Z}\times\mathbb{Z}/H \cong \mathbb{Z}$.



3. $G = \mathbb{R}\times\mathbb{R} = \mathbb{R}^2$, $H = \{(x,0) : x \in \mathbb{R}\} = $ x-axis

   cosets of H in $\mathbb{R}^2$ are lines parallel to x-axis

   $\mathbb{R}\times\mathbb{R}/H \cong \mathbb{R}$, choose coset reps to be y-axis