Homework 1

1. (10 points) Let S and T be sets and let $f : S \to T$ be a function. Prove the following statements:

   (a) If U is a set and $g : T \to U$ is a function such that $g \circ f$ is injective, the f is also injective.

   (b) If R is a set and $h : R \to S$ is a function such that $f \circ h$ is surjective then f is also surjective.

   (c) Show that if $g, h : T \to S$ are functions satisfying $g \circ f = id_S$ and $f \circ h = id_T$, then f is bijective and $g = h = f^{-1}$. Hint: use parts (a) and (b).

2. (5 points) Let S be a set with 3 elements. How many functions are there from S to S? How many bijections are there from S to S? What if S has 4 elements? What if S has n elements ($n \in \mathbb{N}$)? Explain.

3. (10 points) Equivalence relations. Let $\sim$ be an equivalence relation on a set S. For $x \in S$, $[x] = \{y \in S : x \sim y\}$ denotes the equivalence class containing x.

   (a) Prove that for all $x \in S$, $[x] \neq \emptyset$.

   (b) Prove that $\bigcup_{x \in S} [x] = S$.

   (c) Prove that if $y \in [x]$, then $[y] = [x]$ and if $y \notin [x]$, then $[x] \cap [y] = \emptyset$.

   (d) A **partition** of a set S is a collection of subsets of S: $\{S_i \subset S : i \in I\}$, where I is an indexing set, such that $\bigcup_{i \in I} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. Explain why the equivalence classes of $\sim$ partition S.

   (e) Prove that every partition of a set S can be realized as the equivalence classes of a unique equivalence relation.

4. (5 points) Show by way of an example that the binary operation given by subtraction on $\mathbb{Z}$:

$$\begin{aligned} - : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto a - b \end{aligned}$$

   is not associative.

5. (10 points) Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(x) = 3x - 1$.

   (a) Show that f is bijective and compute f^{-1}.

   (b) Find a binary operation $*$ on $\mathbb{Q}$ such that $f : (\mathbb{Q}, +) \to (\mathbb{Q}, *)$ is an isomorphism.

   (c) Find a binary operation $*$ on $\mathbb{Q}$ such that $f : (\mathbb{Q}, *) \to (\mathbb{Q}, +)$ is an isomorphism.

6. (5 points) Let $(G, \cdot)$ be a group. Prove that for all $a \in G$, $(a^{-1})^{-1} = a$.

7. (10 points) Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : a, b, c \in \mathbb{R}, ad \neq 0 \right\}$$

and consider G with the binary operation given by matrix multiplication. Prove that G with this binary operation is a group. Is G abelian? If so prove it. If not, give an example of two elements of G that do not commute.

8. (5 points) Prove that the special linear group,

$$\mathrm{SL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det(A) = 1\}$$

is a subgroup of the general linear group, $\mathrm{GL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det(A) \neq 0\}$.

9. (10 points) Let $f : G \to H$ be a group homomorphism. Show that the image of f is a subgroup of H. That is, show that the subset

$$\mathrm{im}(f) = \{h \in H : \exists g \in G \text{ with } f(g) = h\} \subset H$$

is a subgroup of H.

10. (10 points) Write down all the subgroups of Sym(3). What are their sizes?

11. (extra credit) (10 points) For which n is the set of nonzero equivalence classes modulo n with binary operation given by multiplication, $(\mathbb{Z}/n\mathbb{Z} - \{0 + n\mathbb{Z}\}, \cdot)$, a group? Prove that your answer is correct.

12. **Question:** How difficult was this homework? Feel free to elaborate on the amount of time spent, the harder questions, and the easier questions.