Homework 2

1. (10 points) What are all the generators of $(\mathbb{Z}/35\mathbb{Z}, +)$? What are all the subgroups of $(\mathbb{Z}/35\mathbb{Z}, +)$? Draw a graph with vertices the subgroups of $\mathbb{Z}/35\mathbb{Z}$, and edges representing containment relations between subgroups.

2. (10 points) Orders of products.

   (a) Let a and b be elements of a group G that commute (so $ab = ba$). Let $m = o(a)$ and $n = o(b)$. Show that $o(ab)$ divides the least common multiple of m and n, $\mathrm{lcm}(m, n)$. Give an example of a group G, and elements $a, b \in G$ such that $o(ab) < \mathrm{lcm}(o(a), o(b))$. Hint: Division with remainder.

   (b) Let a and b be elements of a group G that commute (so $ab = ba$). Let $m = o(a)$ and $n = o(b)$, and now assume that $\gcd(m, n) = 1$. Show that $o(ab) = mn$.

   (c) Let m, n be coprime positive integers (means $\gcd(m, n) = 1$). Use part (b) to show that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ is cyclic. What cyclic group is $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ then isomorphic to?

3. (5 points) Let $f : G \to H$ be an isomorphism from a group G to a group H, and let $a \in G$. Prove that the order of a is the same as the order of $f(a)$.

4. (10 points) $Q_8, D_8, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are all groups of order 8. Show that they are pairwise non-isomorphic. (Here, Q_8 denotes the quaternions and D_8 is the dihedral group of order 8.) Hint: Consider orders of elements of each group.

5. (10 points) Let $G = \langle a \rangle$ be a cyclic group of order n. Show that, for every divisor d of n, there is a unique subgroup of G of order d.

6. (10 points) Let G be a group and let $H \subset G$ be a subgroup. The set of right cosets of H in G is denoted $H\backslash G$. Show that $|G/H| = |H\backslash G|$. Hint: Show that the function

$$\begin{array}{ccc} G/H & \longrightarrow & H\backslash G \\ aH & \longmapsto & Ha^{-1} \end{array}$$

   is a bijection.

   This problem shows that the index of a subgroup could be defined as the number of left or right cosets and we would get the same definition.

7. (10 points) The quaternions. Here are three groups: The first group is given by generators and relations:

$$Q_{8,1} = \langle a, b | a^4 = 1, b^2 = a^2, ba = a^3 b \rangle$$

You may assume that $|Q_{8,1}| = 8$. Show that as a set, $Q_{8,1} = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

The second group is given as a set with an explicit binary operation:

$$Q_{8,2} = \{1, i, j, k, -1, -i, -j, -k\}$$

The binary operation is the following relations

$$-1 \cdot 1 = -1, -1 \cdot i = -i, -1 \cdot j = -j, -1 \cdot k = -k$$
$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$$
$$i^2 = j^2 = k^2 = -1$$

along with the rule that -1 commutes with everything and $(-1)^2 = 1$.

The third group is given as a the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by the elements

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Here $i = \sqrt{-1}$ and $\mathbb{C}$ is the complex numbers. We'll denote this group as $Q_{8,3} = \langle A, B \rangle \subset \mathrm{GL}_2(\mathbb{C})$. You may assume that $|Q_{8,3}| = 8$ (convince yourself why this is true).

Prove that all three of these groups are isomorphic to each other. The abstract group that all three of these groups represent is called the quaternions and denoted Q_8.

8. (10 points) Consider the group $Sym(6)$.

   (a) Let σ be the permutation

   $$1 \mapsto 3; 2 \mapsto 5; 3 \mapsto 6; 4 \mapsto 4; 5 \mapsto 2; 6 \mapsto 1$$

   Find the cycle decomposition of σ.

   (b) With the notation of (a), find the cycle decomposition of σ^{-1}.

   (c) Consider the 4 transpositions:

   $$\sigma_1 = (12), \sigma_2 = (23), \sigma_3 = (34), \text{ and } \sigma_4 = (45)$$

   Find the cycle decomposition of $\tau = \sigma_1\sigma_2\sigma_3\sigma_4$, τ^2, τ^3, τ^4, and τ^5.

9. (5 points) Let $G = Sym(3)$ and let $H = \langle (12) \rangle = \{1, (12)\}$. Show that the set of left cosets of H in G is not the same as the set of right cosets of H in G.

10. (extra credit, 10 points) Let $O_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) : A^t = A^{-1}\}$ be the orthogonal group of 2 by 2 matrices (A^t denotes the transpose of the matrix A).

    (a) Show that for $A \in O_2(\mathbb{R})$, $\det(A) = 1$ or -1.

    (b) Show that if $\det(A) = 1$, then A is a rotation of $\mathbb{R}^2$.

    (c) Show that if $\det(A) = -1$, then A is a reflection of $\mathbb{R}^2$. Hint: Consider the eigenvectors of A and show that they are perpendicular.