Homework 3

1. (20 points) Let G be a group and $H \subset G$ a subgroup. Given $g \in G$, let

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

Prove that gHg^{-1} is a subgroup of G. Let $\mathrm{Sub}(G)$ denote the set of all subgroups of G. Prove that this construction defines an action of G on $\mathrm{Sub}(G)$.

2. Let M and N be normal subgroups of a group G.

   (a) (5 points) Show that $N \cap M$ is a normal subgroup of G.

   (b) (15 points) Assume that G/M and G/N are abelian. Show that $G/(M \cap N)$ is abelian. Hint: Show that $f : G \to G/M \times G/N$, $f(g) = (gM, gN)$ is a group homomorphism and use the first isomorphism theorem.

3. Let

$$\mathrm{GL}_2^+ = \{A \in \mathrm{GL}_2(\mathbb{R}) : \det(A) > 0\} \subset \mathrm{GL}_2(\mathbb{R})$$

and let

$$H = \left\{ A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\} \subset \mathrm{GL}_2(\mathbb{R})$$

   (a) (10 points) Prove that $\mathrm{GL}_2^+(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$. What group is the quotient group, $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2^+(\mathbb{R})$? You must justify your answer to the second part. Hint for the quotient $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2^+(\mathbb{R})$: How many left cosets of $\mathrm{GL}_2^+(\mathbb{R})$ are there in $\mathrm{GL}_2(\mathbb{R})$?

   (b) (10 points) Prove that H is a normal subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ and show that $\mathrm{GL}_2^+(\mathbb{R})/H \cong \mathrm{SL}_2(\mathbb{R})$. Hint: Consider the function

$$\begin{aligned} \varphi : \mathrm{GL}_2^+(\mathbb{R}) &\longrightarrow \mathrm{SL}_2(\mathbb{R}) \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{1}{\sqrt{\det(A)}} A = \begin{pmatrix} \frac{a}{\sqrt{ad-bc}} & \frac{b}{\sqrt{ad-bc}} \\ \frac{c}{\sqrt{ad-bc}} & \frac{d}{\sqrt{ad-bc}} \end{pmatrix} \end{aligned}$$

   where $\sqrt{\det(A)} = \sqrt{ad - bc}$ is the positive square root of the determinant of A. Show that φ is a group homomorphism and then use the first isomorphism theorem.

4. Let G be a group and let H be a subgroup of G. Define the map $*$ from $G \times G/H$ to G/H by

$$\begin{aligned} * : G \times G/H &\longrightarrow G/H \\ (a, bH) &\longmapsto abH \end{aligned}$$

(a) (10 points) Show that $*$ is a well defined function, and that $*$ defines a group action of G on G/H.

(b) (10 points) Let $\rho : G \to \mathrm{Sym}(G/H)$ be the homomorphism corresponding to $*$. Show that $\ker(\rho) \subset H$ and that $\ker(\rho) = H$ if and only if H is a normal subgroup of G.

5. Extra credit (10 points): It may be helpful to do this problem in conjunction with problem 1. Determine all possible subgroups of Sym(3), as in homework 1. Give justification for why your list of subgroups is all the subgroups of Sym(3) and why there is no overlap in your list. This should be easier than it was on homework 1 because we have more tools (theorems) at our disposal.

   Now let Sym(3) act on Sub(Sym(3)) as in question 1. Describe the orbits of this action. Calculate the stabiliser subgroups of $\langle(12)\rangle$ and $\langle(123)\rangle$.

6. Extra credit (10 points): Let D_8 be the dihedral group with 8 elements, so

$$D_8 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$$

   Prove that $\langle\tau\rangle$ is a normal subgroup of $\langle\sigma^2, \tau\rangle$, and $\langle\sigma^2, \tau\rangle$ is a normal subgroup of D_8, but that $\langle\tau\rangle$ is not a normal subgroup of D_8. Conclude that if H is a normal subgroup of K and K is a normal subgroup of G, that does <u>not</u> imply that H is a normal subgroup of G.

7. Extra credit (10 points): Let D_{2n} be the dihedral group of order $2n$, so D_{2n} is the group of symmetries of a regular n-gon centered at the origin. Recall that D_{2n} has group presentation

$$D_{2n} = \langle \sigma, \tau | \sigma^n = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^{n-1} \rangle$$

   where σ is rotation by $360/n$ degrees counterclockwise and τ is any reflection that sends the n-gon to itself.

   Determine the conjugacy classes of D_{2n}. How many are there?